Bachelor of Computer Applications
(BCA)

Elementary Mathematics
(DBCACO103T24)

Self-Learning Material
(SEM 1)

Jaipur National University
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PREFACE

Mathematics plays a crucial role in the curriculum of a Bachelor of Computer Applications (BCA) program. It provides a foundation for various concepts and principles that are essential in the field of computer science. In this introduction to mathematics in BCA, we will explore the key areas of mathematics covered in the program and their relevance to computer science.

The book covers topics like Discrete Mathematics, Calculus, Linear Algebra, Probability and statistics

By studying mathematics in BCA, students extend critical thoughts, problem-solving, and analytical skills that are essential for achievement in the field of computer science. Mathematics provide a solid foundation for understanding the theoretical underpinnings of computer science and enables students to tackle complex computational problems.

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UNIT : 1

MATRIX

Learning Objectives:

- To understand Revision of equations reducible to quadratic form.
- To understand the concept of accounting
- To understand the Users of Accounting Information

Structure:

1.1 Revision of equations reducible to quadratic form Simultaneous equations (linear and quadratic) up to 2 variables only.

Revision of Equations Reducible to Quadratic Form:

Quadratic equations are equations in the form $ax^2 + bx + c = 0$, where a, b, and c are constants. However, it is occasionally possible to simplify equations that are not initially quadratic by putting them in quadratic form. When solving equations with square roots or variables raised to powers greater than two, this method is especially helpful.

To reduce an equation to quadratic form, you can use substitution or other algebraic techniques. Here's an example:

Example:

Consider the equation $2(3x - 1)^2 - 5(3x - 1) = 0$. To reduce this equation to quadratic form, we can make a substitution. Let's substitute $y = 3x - 1$.

By substituting y into the equation, we get:

$2y^2 - 5y = 0$

This equation is now in quadratic form, and we can solve it using various methods such as factoring, completing the square, or via the quadratic formula. After finding the solutions for y, we replace the values back into the original substitution to find the values of x.

Simultaneous Equations (Linear and Quadratic) up to 2 Variables:

Simultaneous equations involve two or more equations with the same variables. When one of the equations is linear (involving variables to the power of 1) and the other is quadratic (involving variables to the power of 2), solving the system of equations requires finding values that satisfy both equations simultaneously.

There are several techniques to solve simultaneous equations, including substitution, elimination, and graphical methods.

Example:

Consider the equations:

Equation (1): $2x + y = 5$

Equation (2): $x^2 + y = 4$

We have two options for solving this system: the substitution approach and the elimination method..

Using the substitution method:

1. Solve Equation (1) for one variable. Let's solve for y: $y = 5 - 2x$.
2. Substitute this expression for y in Equation (2): $x^2 + (5 - 2x) = 4$.
3. Simplify and solve the resulting quadratic equation.
4. Replace the obtained values of$x$ back into Equation(1) to find the corresponding value(s) of y.

Using the elimination method:

1. Multiply Equation (1) by a suitable factor to make the coefficients of y the same in both equations. In this case, multiplying Equation (1) by 1 gives: $2x + y = 5$.
2. Subtract Equation (1) from Equation (2) to eliminate y.
3. Simplify and solve the resulting linear equation for x.
4. Replace the obtained values of x back into Equation(1) to find the corresponding values of y.

1.2 Determinants and their six main properties

Below are some basic characteristics of determinants:

If I_n is the identity Matrix of the order $m \times m$, then $det(I)$ is equal to1.

If the Matrix X^T is the transpose of Matrix$X$, then $det\ (X^T) = det\ (X)$.

If Matrix X^{-1} is the inverse of Matrix$X$, then $det\ (X^{-1})\ =\ 1det(X)$
$=\ det(X)^{-1}$.

In the event where X and Y, two square matrices, have the same size, then $det\ (XY)\ =\ det\ (X)\ det\ (Y)$.

$det\ (CX)\ =\ Cadet\ (X)$if Matrix X maintains size $a \times a$ and$C$ is a constant.

The following equation applies if A, B, and C are three equal-sized positive semidefinite matrices. Additionally, the corollary $det\ (A + B)\ \geq det(A)\ +\ det\ (B)$for A, B, and $C \geq\ 0det\ (A + B + C)\ +\ detC \geq det\ (A + B)\ +\ det\ (B + C)$ holds.

The modulo in a triangular matrix is the multiplication of the diagonal components. If every member in a matrix is equal to zero, the matrix's determinant is zero.

Crucial Characteristics of Determinants

Ten often used and significant features of determinants. These characteristics facilitate computations and aid in the resolution of several issues. Below is a description of each of the ten significant characteristics of determinants.

- Reflection Property
- All-zero Property
- Proportionality
- Switching property
- Factor property
- Scalar multiple properties
- Sum property
- Triangle property
- Determinant of cofactor Matrix
- Property of Invariance

We go into further depth on each of these attributes below:

1. **Reflection Property**

Determinants reflection property states that when rows are converted into columns and vice versa, Determinants remain unchanged.

2. **All- Zero Property**

If all of the terms in the rows and columns are zero, then the Determinants will also be zero.

3. **Proportionality (Repetition Property)**

The Determinant is equal to zero if every term in a row or column is comparable to the column of another row (or column).

4. **Switching Property**

Any two rows or columns of the determinant that are switched around alter the sign of the result.

5. **Factor Property**

(x -α) is regarded as a factor of Δ if a determinant Δ reaches 0 while taking the value of x = α into consideration.

6. **Scalar Multiple Property**

A comparable constant multiplies the determinant if every element in a row or columns is multiply by a non-zero constant.

7. **Sum Property**

$$\begin{vmatrix} j_1+k_1 & l_1 & m_1 \\ j_2+k_2 & l_2 & m_2 \\ j_3+k_3 & l_3 & m_3 \end{vmatrix} = \begin{vmatrix} j_1 & l_1 & m_1 \\ j_2 & l_2 & m_2 \\ j_3 & l_3 & m_3 \end{vmatrix} + \begin{vmatrix} +k_1 & l_1 & m_1 \\ +k_2 & l_2 & m_2 \\ +k_3 & l_3 & m_3 \end{vmatrix}$$

8. **Triangle Property**

A Determinant is equal to the product of diagonal terms if every term, whether above or below the main diagonal, consists of zeroes, i.e.,

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & y_2 & y_3 \\ 0 & 0 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & 0 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & z_3 \end{vmatrix} = X_1 Y_2 Z_3$$

9. **Determinant of Cofactor Matrix**

$$\Delta = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & y_{22} & y_{23} \\ x_{31} & x_{32} & z_{33} \end{vmatrix} then\Delta_1 = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix} = \Delta^2$$

In the above modulo of the co-factor Matrix, Z_{ij} depicts the co-factor of the element x_{ij} in Δ

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10. Property of Invariance

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix}$$

It suggests that Determinant does not alter when the phrase $C_i \to C_i + \alpha C_j + \beta C_k$ is used.

where$j$ and k are not equal to i, or when j and k are not equal to i in a mathematical operation of the expression $R_i \to R_i + \alpha R_j + \beta R_k$.

1.3 Solutions of simultaneous equations by Cramer's rules

Cramer's Rule is a technique utilized to crack systems of simultaneous linear equations using determinants. It presents a systematic approach to find out the matchless result for each variable in the system. Let's explore how Cramer's Rule works:

Consider linear equations:

$$x\square x + y\square y + z\square z = d\square$$
$$x\square x + y\square y + z\square z = d\square$$
$$x\square x + y\square y + z\square z = d\square$$

To solve this system using Cramer's Rule:

Compute the mod of the coefficient matrix, often denoted as D, which is specified by:

$$D = \begin{vmatrix} x\square & y\square & z\square \\ x\square & y\square & z\square \\ x\square & y\square & z\square \end{vmatrix}$$

Compute the determinants of the matrices obtained by replacing each column of the coefficient matrix with the column of constants, denoted as D_1 , D_2 , and D_3 . These determinants represent the solutions for x, y, and z, respectively:

$$D_1 = \begin{vmatrix} d\square & y\square & z\square \\ d\square & y\square & z\square \\ d\square & y\square & z\square \end{vmatrix}$$

$$D_2 = \begin{vmatrix} x\square & d\square & z\square \\ x\square & d\square & z\square \\ x\square & d\square & z\square \end{vmatrix}$$

$$D_3 = \begin{vmatrix} x\square & y\square & d\square \\ x\square & y\square & d\square \\ x\square & y\square & d\square \end{vmatrix}$$

Calculate the values of x, y, and z by dividing the determinants D_1 , D_2 , and D_3 by the determinant D:

$$x = D\square \,/\, D$$

$$y = D\square \,/\, D$$

$$z = D\square \,/\, D$$

Replace the computed values of x, y, and z back into the original equations to verify the solution.

Cramer's Rule provides a method to find out the unique solutions for each variable in a system of linear equations without the need for row operations or matrix inverses. Yet, it is significant to note down that Cramer's Rule is not efficient for large systems due to the computational complexity involved in calculating determinants.

1.4 Users of Accounting Information

In mathematics, special matrices are matrices that possess certain distinctive properties or exhibit specific characteristics. These matrices often have special names and play important roles in various mathematical applications. Here are definitions of some commonly encountered special matrices:

1. **Identity Matrix (I):**

The identity matrix, often denoted as I or I□, is a square matrix with ones on the chief diagonal and zeros in another place. Specifically, the element in the i^{t}□ row and j^{t}□ column is 1 if $i = j$, and 0 otherwise. The identity matrix serves as the multiplicative identity for matrix multiplication, similar to how 1 is the multiplicative identity for real numbers.

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 \\ . & . & 1 & & & . \\ . & . & & 1 & & . \\ . & . & & & 1 & . \\ 0 & 0 & . & . & . & 1 \end{bmatrix}_{n\times n}$$

2. **Zero Matrix (O):**

The zero matrix, denoted as O or O□ □ , is a matrix in which all elements are zeros.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. **Diagonal Matrix:**

A diagonal matrix is one in which there are zeros for each non-diagonal member. A rectangular array of integers organized in rows and columns is called a matrix.

$$\text{diagonal\_matrix}(\{1,2,3\}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{Calc}$$

$$\text{diagonal\_matrix}(1..20..4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 & 17 \end{pmatrix}$$

$$\text{diagonal\_matrix}([3,3,2,4,1]) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Symmetric Matrix:

A square matrix that is equal to its transpose matrix is said to be symmetric. Any given transpose matrix of matrix A may be expressed as A^T. Thus, the requirement $A = A^T$is satisfied by a symmetric matrix$A$.

5. Skew-Symmetric Matrix:

A square matrix that is equal to its transpose matrix's negative is known as a skew symmetric matrix in mathematics. The transpose matrix for every square matrix A is denoted by A^T. Therefore, the representation of a skew-symmetric or antisymmetric matrix $A$is $A = -A^T$.

Symmetric

$$A^T = A$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

Skew-symmetric

$$A^T = -A$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

6. Orthogonal Matrix:

If the transpose of a square matrix A is equal to its inverse, then matrix A is an orthogonal. For example, $A^T = A^{-1}$, where A^{-1} is the inverse of A and A^T is the transpose of A.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = A^{-1}$$

$$AA^T = A^TA = I$$

1.5 Arithmetic operation on matrices

Arithmetic operations on matrices involve various mathematical operations such as addition, subtraction, scalar multiplication, matrix transpose, adjoint, and matrix inverse. These operations allow us to manipulate matrices and derive new matrices based on specific rules. Let's explore each operation in more detail:

1) Addition:

If we have two matrices A and B of the same size m x n, their sum C, denoted as $C = A + B$, is obtained by adding corresponding elements: $C_{ij} = A_{ij} + B_{ij}$, here i and j stand for the row and column.

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ -10 & -11 & -12 \end{bmatrix} \quad \text{and } \mathbf{B} = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 12 & 1 \\ 19 & 9 & 3 \end{bmatrix}$$

$$A + \mathbf{B} = \begin{bmatrix} 3 - 1 & 4 - 2 & 5 - 3 \\ 6 + 4 & 7 + 12 & 8 + 1 \\ -10 + 19 & -11 + 9 & -12 + 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 10 & 19 & 9 \\ 9 & -2 & -9 \end{bmatrix}$$

2) Subtraction:

If we have matrices A and B of the same size mxn, their difference C, denoted as $C = A - B$, is obtained by subtracting corresponding elements: $C_{ij} = A_{ij} - B_{ij}$.

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ 4 & 5 \\ 3 & 7 \end{bmatrix}} - \underset{B}{\begin{bmatrix} 2 & -6 \\ 8 & 4 \\ 5 & -2 \end{bmatrix}} = \underset{A - B}{\begin{bmatrix} 0 - 2 & 1 - (-6) \\ 4 - 8 & 5 - 4 \\ 3 - 5 & 7 - (-2) \end{bmatrix}}$$

3) Scalar Multiplication:

If we have a matrix A of size $m \times n$ and a scalar k, the scalar multiplication is denoted as $C = kA$.

Example:

$$A = \begin{bmatrix} 3 & 7 \\ 9 & 10 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 \cdot 3 & 2 \cdot 7 \\ 2 \cdot 9 & 2 \cdot 10 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ 18 & 20 \end{bmatrix}$$

4) Transpose Matrix:

A transposed matrix is the original matrix reversed. By flipping the rows and columns of a matrix, we can transpose it. We signify transposition of matrix A by A^T.

A	A^T
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
$[5]$	$[5]$
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$	$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

5) Adjoint (or Adjugate) Matrix:

The adjoint of a matrix B is the transpose of the co-factor matrix of B. The adjoint of a square matrix B is denoted by $adjB$. Let $B = [b_{ij}]$ be a square matrix of order n.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

6) Inverse Matrix:

To find the inverse of A, several methods can be used, such as Gauss-Jordan elimination, matrix of cofactors, or using the adjoint and determinant. Not all matrices have inverses, and for a matrix to be invertible, its modulo must not be zero.

Inverse of a 2 x 2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times \text{Adj } A$$

where,

$$|A| = ad - bc$$

$$\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of a 3 x 3 Matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \begin{bmatrix} (ei-fh) & -(bi-ch) & (bf-ce) \\ -(di-fg) & -(ai-cg) & -(af-cd) \\ (dh-eg) & -(ah-bg) & (ae-bd) \end{bmatrix}$$

where,

$$|B| = a(ei-fh) - b(di-fg) + (dh-eg)$$

1.6 Solution of Simultaneous Equations Using Matrices

Step-1 Write the equations in matrix form: Convert the system of simultaneous equations into matrix form. For example, consider the system:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

This can be represented in matrix form as AX=B, where:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} e \\ f \end{bmatrix}$$

Step-2 Find the inverse of matrix A: If the matrix A is invertible, find its inverse A^{-1} .

Step-3 Compute the solution: Multiply both sides of the equation $AX = B$ A^{-1} to solve for X: $X = A^{-1}B$.

Step-4 Verify the solution: Plug the values of x and y back into the original equations to ensure they satisfy all equations.

Example. Solve the simultaneous equations

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

Solution. We have already seen these equations in matrix form: $\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

We need to calculate the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$.

$$A^{-1} = \frac{1}{(1)(-5) - (2)(3)} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix}$$

Then X is given by

$$\begin{aligned} X = A^{-1}B &= -\frac{1}{11} \begin{pmatrix} -5 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -22 \\ -11 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Hence $x = 2$, $y = 1$ is the solution of the simultaneous equations.

Example. Solve the simultaneous equations

$$\begin{aligned} 2x + 4y &= 2 \\ -3x + y &= 11 \end{aligned}$$

Solution. In matrix form: $\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$.

We need to calculate the inverse of $A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$.

$$A^{-1} = \frac{1}{(2)(1) - (4)(-3)} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$$

Then X is given by

$$\begin{aligned} X = A^{-1}B &= \frac{1}{14} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 11 \end{pmatrix} \\ &= \frac{1}{14} \begin{pmatrix} -42 \\ 28 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 2 \end{pmatrix} \end{aligned}$$

1.7 Summary

There are many algebraic topics that involve dealing with matrices and solving problems. We begin by revising equations reducible to quadratic form, which entails converting equations into quadratic equations and solving themutilising techniques like factoring or the quadratic formula. We also research simultaneous equations with a maximum of two variables, both linear and quadratic. This enables us to resolve equation problems with several variables.

We investigate determinants and their characteristics before moving on to matrices. When it comes to square matrices, determinants are numerical values that are connected to crucial characteristics including linearity, multiplicative property, and inverse determinant. We also study Cramer's laws, which offer a way to resolve linear equation systems using determinants.

Next, we delve into matrices and their operations. Special matrices, such as unit matrices, singular matrices, and diagonal matrices, are defined. These matrices have distinct qualities that make them helpful in a large range of applications. Arithmetic operations on matrices, such as adding, subtracting, and scalar multiplicating, enable us to handle and calculate with matrices.

1.8 Keywords

- **Quadratic Equations: Quadratic equations are polynomial equations of the second degree, typically** in the form of $ax^2 + bx + c = 0$, here a, b, and c are invariants and$x$ represent the variable. Square roots are frequently used in the solutions to quadratic equations, which can be cracked by factoring, completing the square, or using the quadratic formula. They are essential in various fields, including physics, engineering, and finance.
- **Simultaneous Equations:** Simultaneous equations refer to a system of equations that are cracked together to find the values of multiple variables. These equations represent relationships between different variables and are typically written in the form of equations involving multiple variables. Simultaneous equations can be linear or quadratic and are used to solve problems involving multiple unknowns. They have applications in various fields, such as physics, economics, and optimization.

1.9 Self-Assessment Questions

1. What are the steps concerned in solving a quadratic equation that can be reduced to quadratic form? Explain the methods used to solve such equations.

2. How do you solve a system of simultaneous equations involving linear and quadratic equations with up to two variables? Provide an example.
3. What are the six important properties of determinants in linear algebra? Explain each property briefly.
4. Explain Cramer's rules for solving simultaneous equations. When are Cramer's rules applicable?
5. Define special matrices and provide examples of each type.

1.10 Case Study

Product A and Product B are the two kinds of products manufactured by XYZ Manufacturing Company. The business wishes to reduce production costs by calculating the optimal amounts of each product. Raw material availability, labour costs, and equipment utilisation are all aspects of the manufacturing process. The firm has gathered information on the costs and restrictions of manufacturing both products. As a data analyst, you use determinants and Cramer's rules to determine the ideal production quantities that minimise total production costs.

Questions:

1. Suppose the given system of equations represents the production costs and constraints for Product A and Product B: $2A + 3B = 240$ and $4A - B = 160$
   Calculate the determinant of the coefficient matrix and verify if Cramer's rules can be applied in this case.
2. Discuss the advantages and disadvantages of using determinants and Cramer's rules compared to other methods of solving simultaneous equations in terms of accuracy, computational complexity, and applicability to real-world scenarios.
3. Explain how Cramer's rules can be applied to find the optimal production quantities of Product A and Product B.

1.11 References

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UNIT : 2

TRIGONOMETRY

Learning Objectives:

- To understand Trigonometry
- To understand the concept of accounting
- To understand the Users of Accounting Information

Structure:

2.1 Trigonometry: Revision of angle measurement

2.2 T-ratios addition, subtraction and transformation formulae

2.3 T-ratio of multiple and allied angles

2.4 Summary

2.5 Keywords

2.6 Self-Assessment Questions

2.7 Case Study

2.8 References

2.1 Trigonometry: Revision of angle measurement

The study of the relationships between angles and triangle sides is known as trigonometry.

It includes the study of angles, their measurements, and their qualities. There are numerous crucial principles to comprehend while reviewing angle measuring in trigonometry.

Degree Measurement: The most widely used unit of measurement for angles is degrees. A circle has 360 equal segments, and a degree is the name given to each segment. The symbol for degrees is °.

Radian Measurement: Radians are another unit used to measure angles in trigonometry. The angle subtended in the center of a circle by an arc whose length is equal to the circle's radius is called a radian. 2π radians make up a full circle, where π (pi) is a mathematical constant that is roughly equivalent to 3.14159.

Converting between Degrees and Radians: Multiply the degree measure by $\pi/180$ to convert from degrees to radians. You multiply the radian measurement by $180/\pi$ to convert radians to degrees.

Special Angles: In trigonometry, certain angles have special properties and measurements. These include the 30°-60°-90° triangle, the 45°-45°-90° triangle, and the unit circle, which is a circle with a radius of 1 unit.

Angle Relationships: Angles have reciprocal relationships, such as complementary angles, supplementary angles, and vertical angles. When two lines cross, complementary angles sum up to 90 degrees, supplementary angles add up to 180 degrees, and vertical angles are opposite each other.

2.2 T-ratios addition, subtraction and transformation formulae

Trigonometric ratios, or T-ratios, are the trigonometric ratios of the side lengths of a right triangle. Sine (sin), cosine (cos), and tangent (tan) are the three fundamental T-ratios. For manipulating and simplifying trigonometric formulas using these T-ratios, addition, subtraction, and transformation formulae are useful.

Addition and Subtraction Formulae:

- Sum Formula:
  - $Sin: \sin(A+B) = sinA.cosB + cosA.sinB$
  - $cosine: \cos(A+B) = cosA.cosB - sinA.sinB$
  - $Tangent: \tan(A+B) = \frac{(tanA + tanB)}{(1 - tanA.tanB)}$

- Difference Formula:
  - $Sin: \sin(A-B) = sinA.cosB - cosA.sinB$
  - $cosine: \cos(A-B) = cosA.cosB + sinA.sinB$
  - $Tangent: \tan(A-B) = \frac{(tanA - tanB)}{(1 + tanA.tanB)}$

These formulas let us know the T-ratios of the individual angles and then use them to get the sine, cosine, and tangent of the sum or difference of two angles.

Transformation Formulae:

Double-Angle Formulae:

Sine: sin(2A) = 2sin(A) . cos(A)

Cosine: cos(2A) = $cos^2(2A) - sin^2(2A) = 2cos^2(2A) - 1 = 1 - 2sin^2(2A)$.

Tangent: tan(2A) = $(2\tan(A))/(1 - tan^2(2A))$.

Half-Angle Formulae:

- Sine: $\sin(A/2) = \frac{\pm\sqrt{(1-\cos(A))}}{2}$
- Cosine: $\cos\,(\mathrm{A}/2) = \frac{\pm\sqrt{(1+\cos(A))}}{2}$
- Tangent: $\tan(A/2) = \pm\frac{\sqrt{(1-\cos(A))}}{\sqrt{(1+\cos(A))}}$

These formulas let us represent the sine, cosine, and tangent of an angle in terms of half the angle, making trigonometric expressions easier to evaluate.

We may simplify and modify trigonometric formulas using T-ratios by using these addition, subtraction, and transformation formulae. These tools are extremely useful for solving trigonometric equations, assessing complicated trigonometric functions, and addressing issues in physics, engineering, and mathematics.

2.3 T-ratio of multiple and allied angles

- **T-ratio of multiple angles**

For angles that are multiples of a specific angle in trigonometry, the values of the trigonometric functions are known as the T-ratios (trigonometric ratios) of multiple angles.

These ratios enable us to identify the T-ratios of related angles, making it easier to calculate and solve trigonometric issues.

The T-ratios of several angles can be described in terms of the original angle's T-ratios. Here are some important T-ratios for multiple angles:

Double Angle Formulas:

- $\sin(2\theta) = 2sin\theta.cos\theta$
- $\cos(2\theta) = cos^2\theta - sin^2\theta = 2cos^2\theta - 1$
- $\tan(2\theta) = \frac{2tan\theta}{1-tan^2\theta}$

Triple Angle Formulas:

- $\sin(3\theta) = 3sin\theta - 4sin^3\theta$
- $\cos(3\theta) = 4cos^3\theta - 3cos\theta$
- $\tan(3\theta) = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$

These formulae allow us to calculate the T-ratios of angles that are twice or three times the specified angle. We can compute the T-ratios for these many angles by knowing the T-ratio of the original angle.

Multiple angle T-ratios have several uses in trigonometry, including solving trigonometric equations, simplifying trigonometric expressions, and analysing periodic functions. They assist us in comprehending the patterns and relationships that exist between various angles, making it simpler to work with trigonometric functions in a variety of mathematical and scientific situations.

T-ratio of Allied Angles

The T-ratios (trigonometric ratios) of allied angles are the values of trigonometric functions for angles with the same T-ratios but different signs in trigonometry. Because these angles are connected, we may simplify computations and solve trigonometric issues more effectively.
Here are some important T-ratios for allied angles:

Complementary Angles:

Two angles are complementary if their sum is 90 degrees (π/2 radians).
For complementary angles, the cosine and sine of one angle are equivalent, and vice versa.

$$sin\theta = cos(90° - \theta)$$

$$cos\theta = sin(90° - \theta)$$

Supplementary Angles:

Two angles are supplementary if their sum is 180 degrees (π radians).

In the case of supplementary angles, the cosine of one angle equals the negative cosine of the other, and the sine of one angle equals the sine of the other.

$$sin\theta = sin(180° - \theta)$$

$$cos\theta = -cos(180° - \theta)$$

Negative Angles:

A negative angle is formed by rotating clockwise from the positive x-axis.

For negative angles, the sine and tangent functions have the same absolute value but differ in sign, while the cosine function remains unchanged.

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = cos\theta$$

$$\tan(-\theta) = -tan\theta$$

Understanding allied angle T-ratios helps us to identify related angle T-ratios without having to compute them independently. These connections aid in the simplification of trigonometric expressions, the solution of trigonometric equations, and the analysis of periodic functions. We can work with trigonometric functions more effectively in many mathematical and scientific applications if we recognize the patterns and connections between associated angles.

2.4 Summary

Trigonometry is the study of angles and their links to triangle sides. It is critical to grasp degree and radian measurements, conversions between them, special angles, and angle connections while revisiting angle measurement in trigonometry. Degree measurement divides a circle into 360 equal sections, whereas radian measurement measures angles using the radius notion. Special angles with specific trigonometric values, such as 30°, 45°, and 60°, are widely utilised. Angle connections like complementary, supplementary, and vertical angles aid in the solution of trigonometric problems.

T-ratios, often known as trigonometric ratios, are essential in trigonometry. They link a triangle's angles to the lengths of its sides. T-ratio addition and subtraction formulae entail using the sine, cosine, and tangent functions to get the T-ratios of the sum and difference of two angles. By linking the T-ratios of half angles to the T-ratios of the original angles, the transformation formulae aid in determining the T-ratios of half angles. Trigonometric equations can be simplified and evaluated using these formulae.

Understanding the T-ratios of numerous angles is also important in trigonometry. The T-ratios of an angle twice the size of a particular angle are expressed using the double angle formulae.Similarly, triple angle formulae provide T-ratios for an angle three times the size of the provided angle. These formulas make it easier to calculate T-ratios of numerous angles.

Aside from various angles, allied angles are also important in trigonometry. The T-ratios of allied angles are the same, but the sign is different. Complementary angles add up to 90°, and their T-ratios are reciprocal. The total of the supplementary angles is 180°, and their T-ratios have the same absolute value but differ in sign. T-ratios for negative angles have the same absolute value as positive angles but opposite signs.

Understanding trigonometric concepts like angle measurement, T-ratio addition and subtraction, transformation formulae, T-ratios of multiple angles, and allied angles provides a solid foundation for solving trigonometric problems, analysing geometric relationships, and applying trigonometry in fields like physics, engineering, and navigation.

2.5 Keywords

Angle measurement: Angle measurement is the technique of determining the size of an angle. It entails expressing the magnitude of angles using several units of measurement, such as degrees and radians. Understanding angle measurement is essential in trigonometry since it enables exact computations and geometric connection analysis.

T-Ratios: T-ratios, or trigonometric ratios, relate the angles of a triangle to its side lengths. The three basic T-ratios are sine (sin), cosine (cos), and tangent (tan). T-ratios give useful information on triangle geometric features and are often utilised in trigonometric calculations such as estimating side lengths, angle measurements, and solving trigonometric equations. T-ratio addition, subtraction, and transformation formulae extend their value in trigonometry. Understanding T-ratios for various and related angles also enables the simplification and manipulation of trigonometric formulas.

2.6 Self-Assessment Questions

1. What is the difference between degree and radian measurement in trigonometry? Explain the conversion between the two.
2. Using the addition and subtraction formulas for T-ratios, calculate the exact value of sin (75°) in terms of known T-ratios.
3. Apply the transformation formula for T-ratios to find the value of cos(π/12) in terms of known T-ratios.
4. Find the T-ratio of the double angle for an angle θ if sin(θ) = 3/5 and θ is in the first quadrant.
5. Given that $\sin(x) = \frac{1}{2}, \cos(y) = -3/5$, and x and y are allied angles, calculate the value of $\tan(x - y)$ using T-ratio properties.

2.7 Case Study

A renewable energy company intends to put solar panels on a commercial building's rooftop. To maximise energy efficiency, solar panels must be precisely positioned and angled. You are responsible for doing trigonometric calculations as part of the installation crew to find the ideal angles and dimensions for the solar panels. This includes revising angle measurements, T-ratio addition, subtraction, transformation equations, and considering T-ratios of numerous and associated angles.

The dimensions and specifications of the commercial building and the solar panels are as follows:

Building height: 20 meters

Width of the rooftop: 10 meters

Solar panel dimensions: 2 meters by 1 meter

1. Calculate the slope or inclination angle at which the solar panels should be installed to maximize energy absorption.
2. Utilizing trigonometric principles, determine the T-ratios (sine, cosine, and tangent) for the calculated inclination angle.
3. Explain how the addition, subtraction, and transformation formulae can be applied to adjust the solar panel angles based on specific requirements or constraints.

2.8 References

1. Grewal . B.S., "Elementary Engineering Mathematics", Khanna publications 34th Ed., 2020.
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UNIT : 3

COORDINATE GEOMETRY

Learning Objectives:

- To understand Analytical plane geometry
- To understand the Cartesian coordinates
- To understand the distance between two points
- To understand the area of triangle

Structure:

3.1 Cartesian coordinates
3.2 Distance between two points
3.3 T-ratio of multiple and allied angles
3.4 locus of point
3.5 Straight line
3.6 Slope and intercept form
3.7 General equation of first degree
3.8 Summary
3.9 Keywords
3.10 Self-Assessment Questions
3.11 Case Study
3.12 References

3.1 Cartesian coordinates

Analytical plane geometry, or Cartesian coordinates, is a discipline of mathematics that gives a systematic method for studying and analysing geometric structures on a two-dimensional plane. It entails representing points, lines, curves, and forms using a coordinate system known as the Cartesian coordinate system.

An ordered pair (x, y), where x denotes the horizontal location along the x-axis and y denotes the vertical position along the y-axis, represents each point in the plane in Cartesian coordinates. The x-axis and y-axis are perpendicular lines that cross at the origin (0, 0), which acts as the coordinate system's reference point.

Distances, angles, and other geometric features may be precisely measured and represented using the Cartesian coordinate system. It provides a foundation for executing numerous geometric operations and transformations.

Concepts in analytical plane geometry using Cartesian coordinates include:

Distance Formula: This formula can be used to determine the separation between two points, (x_1 , y_1) and (x_2 , y_2).: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This formula is derived from the Pythagorean theorem and allows us to find the length of a line segment between two points.

Slope of a Line: Use this formula to find the slope of a line that passes through two points, $(x_1, y_1) and\ (x_2, y_2)$; $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$The slope indicates the steepness or inclination of the line and provides information about its direction.

Equation of a Line: There are other ways to represent the equation of a line: the point-slope form (y - y_1 = m(x - x_1)), where (x_1 , y_1) is a point on the line, or the slope-intercept form (y = mx + b), where m is the slope and b is the y-intercept. We may analyze and characterize the characteristics of lines in the coordinate plane using these equations.

Midpoint Formula: The midpoint formula can be used to find the midpoint between two points $(x_1, y_1) and\ (x_2, y_2)$: $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$The midpoint (whose coordinates are the average of the x- and y-coordinates) splits the line segment that connects the two locations into two equal halves.

Transformation: Cartesian coordinates allow for various transformations of geometric figures, including translations, reflections, rotations, and dilations. These transformations can be represented algebraically using coordinate formulas and enable us to manipulate and analyse shapes in the coordinate plane.

Cartesian coordinates in the analytical plane Geometry is a useful tool for solving geometric issues, analysing geometric connections, and applying geometry to a variety of subjects like physics, engineering, computer graphics, and others. It serves as a fundamental notion in mathematics and acts as the foundation for future research of higher issues in geometry.

3.2 Distance between two points

Finding the separation between two points in a plane is possible with the distance formula, which is based on the Pythagorean theorem. Think about the two points (x_1 , y_1) and (x_2 , y_2) that have coordinates. The following formula can be used to determine the distance, d, between these points:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The difference between the two points' x-coordinates is represented by (x_2 - x_1) in this formula, while the difference between their y-coordinates is represented by

(y_2 - y_1). The square of the distance between the spots can be found by squaring and adding these discrepancies. The real distance can be found by taking the square root of this total.

For example, Let us take two points, A(2, 3) and B(5, 7), We enter the coordinates into the distance formula to determine the separation between these points:

$d = \sqrt{(5 - 2)^2 + (7 - 3)^2}$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$

Consequently, there are 5 units separating points A and B.

To determine the separation between any two locations in a plane, using the distance formula.It gives a quantitative measure of point separation, which is necessary for many geometric applications. The distance formula is a useful tool in analytical plane geometry for estimating the length of a line segment, computing the dimensions of a shape, or addressing real-world problems involving distance.

3.3 T-ratio of multiple and allied angles

Techniques for Calculating a Triangle's Area

There are three ways to find the area of a triangle. The following discusses the three distinct approaches.

Approach 1: Given the triangle's base and altitude.

The triangle's area, A = bh/2 sq units

where b and h represent the triangle's base and altitude, respectively.

Approach 2: Based on the lengths of its three sides, one can apply Heron's formula to find a triangle's area.

Thus, the equation is used to get the triangle's area.

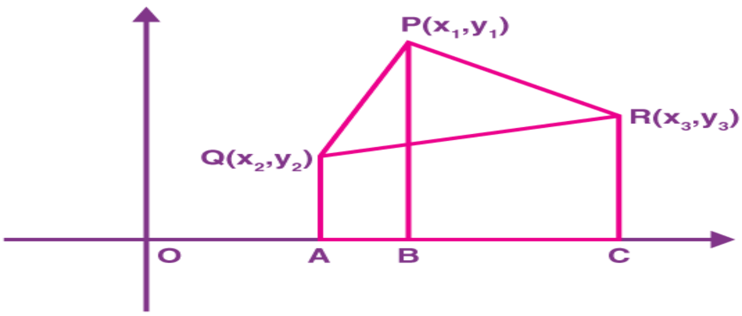
$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

where s is the semi-perimeter and a, b, and c are the triangle's side lengths.

The value of s can be determined by this formula

$$s = \frac{a + b + c}{2}$$

Approach 3: We must first determine the lengths of the triangle's three sides if we know its vertices. You can use the distance formula to find the length. How to calculate a triangle's area when you know its vertices in the coordinate plane. Assume that there is a triangle PQR, with the coordinates P, Q, and R being (x1, y1), (x2, y2), and (x3, y3), respectivly.



Based on the triangle PQR's area in the figure, lines like QA, PB, and RC are drawn perpendicular to the x-axis from Q, P, and R, respectively.

In the coordinate plane, three distinct trapeziums, such as PQAB, PBCR, and QACR, have now formed.

Determine each trapezium's area now.

It follows that the formula for calculating the

$area\ of\ \Delta PQR = (area\ of\ trapezium\ PQAB + area\ of\ trapezium\ PBCR) -$
$(area\ of\ trapezium\ QACR\ )$ — (1)

Determining the Trapezium PQAB's Area

We are aware that the formula for calculating a trapezium's area is

Since Area of a trapezium = $\frac{1}{2}(Sum\ of\ paralle\ sides).(distance\ between\ t\square em)$

Area of trapezium PQAB = $\frac{1}{2}(QA + PB).(AB)$

QA = y_2

PB = y_1

AB = OB – OA = $x_1 - x_2$

Area of trapezium PQAB = $\frac{1}{2}(y_1 + y_2).(x_1 - x_2)$ — (2)

Finding Area of a Trapezium PBCR

Area of trapezium PBCR =$\frac{1}{2}(PB + CR).(BC)$

PB = y_1

CR = y_3

BC = OC – OB =$x_3 - x_1$

Area of trapezium PBCR = $\frac{1}{2}(y_1 + y_3).(x_3 - x_1)$— (3)

Finding Area of a Trapezium QACR

Area of trapezium QACR = (1/2) (QA + CR) × AC$\frac{1}{2}(QA + CR).(AC)$

QA = y_2

CR = y_3

AC = OC – OA = $x_3 - x_2$

Area of trapezium QACR = $\frac{1}{2}(y_3 + y_2).(x_3 - x_2)$—(4)

Substituting (2), (3) and (4) in (1),

Area of ΔPQR = $\frac{1}{2}[\{(y_1 + y_2).(x_1 - x_2)$ + $(y_1 + y_3).(x_3 - x_1)\}$– $\{(y_3 + y_2).(x_3 - x_2)\}]$

$A = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1).x_3(y_1 - y_2)]$

Special Case:

The formula below can be used to get the area of a triangle if the origin is one of its vertices.

Area of a triangle with vertices are (0,0), P(a, b), and Q(c, d) is

$$A = (1/2)[0(b - d) + a(d - 0) + c(0 - b)]$$

$$A = (ad - bc)/2$$

If area of triangle with vertices $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is zero, then $(1/2) [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3(y_1 - y_2)] = 0$ and the points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$are collinear.

3.4 Locus of Points

The collection of all points that meet a given condition or criterion is referred to as the locus of points. It aids in describing the path or region that the points take in response to a certain geometric connection. Equations or criteria that the points must fulfil determine the location of the points.

In mathematics, a locus is a curve or other shape formed by a point, line, or moving surface, or by all the points meeting a particular equation of the connection between the coordinates. All forms, including ellipses, parabolas, hyperbolas, and circles, are defined by the locus as a collection of points.

In real life, you've undoubtedly heard the phrase "location." The word location is the root of the term locus. Anything's locus is determined by it. An object's location or an event that takes place there is described by its locus. For instance, the region is now the center of government opposition.

What is Locus?

A locus is a group of points whose locations are determined by specific restrictions. Take the Southwest range, which has hosted multiple campaigns for independence, as an example. In this sense, the locus is defined as the center of any location.

In mathematics, a locus is a collection of points that is represented by a particular rule, law, or equation.

The Locus of Points

In geometry, the locus of points determines a shape. Assume that a circle is the location of all points that are equidistant from the centre. Similarly, the locus of the points defines various forms such as an ellipse, parabola, hyperbola, and so on.

Important Locus Theorems

In geometry, six significant locus theorems are well-known. These theorems may appear complicated at first glance, yet their concepts are simple. Let us go through the six most significant theorems in depth.

Locus Theorem 1:

The locus is conceptualized as a circle with "p" as its center and "d" as its diameter, located at a specific distance "d" from a point "p."

The region created by all points that are the same distance apart from a single point can be found with the help of this theorem.

Locus Theorem 2:

A pair of parallel lines on either side of "m" at a distance "d" from the line "m" is the definition of the locus at a given distance "d" from the line "m."

This theorem helps find the area generated by each point put at an equal distance from a single line.

Locus Theorem 3:

The loci that are perpendicular bisectors of the line segment that connects two locations, let's say A and B, are considered to be equally spaced from each other. By applying this theorem, one can determine the region made up of all points that are the same distance from points A and B. The perpendicular bisector of line segment AB should be the produced region.

Locus Theorem 4:

A locus is considered to be parallel to both lines if it is equally spaced from them, such as m1 and m2. It should be located midway between them.

The application of this theorem allows one to determine the region comprised of all points that are equally spaced from two parallel lines.

Locus Theorem 5:

The point that lies inside an angle and is equally spaced from its sides is known as the bisector of an angle.

This theorem allows one to determine the area generated by all points equally spaced from both sides of an angle. The region must divide the angle in half.

Locus Theorem 6:

A pair of lines that bisects the angle created by the two intersecting lines, m1 and m2, is defined as the locus that is equidistant from them. The area formed by all points positioned at equal distances from the two crossing lines can be found with the help of this theorem. Two lines that split the generated angle should represent the created area.

3.5 Straight Line

When studying geometric shapes and their attributes, an understanding of the straight line is essential. The shortest path between two places is a straight line, which can be expressed in a number of ways using different mathematical representations.

Slope-Intercept Form:

One common way to describe a straight line is using the slope-intercept form, which is Y = mx + b. In this form, m represents the line's slope, which indicates how steep or incline it is, and b represents the y-intercept, or the point where the line crosses the y-axis.

Point-Slope Form:

The equation $y - y_1 = m(x - x_1)$ provides the point-slope form, which is an additional illustration of a straight line. In this case, m denotes the line's slope and (x_1, y_1) denotes a point on the line.

General Form:

The general form of a straight line equation is $Ax + By + C = 0$, where A, B, and C are constants. This form allows for a more flexible representation of lines, including vertical lines and lines with undefined slopes.

Slope-Intercept Equation:

A straight line' slope and intercept are represented by the equation$y = mx + b$, in which m stands for the line's slope and b for the y-intercept. Finding the link between the variables x and y is one usage for this form.

Properties of Straight Lines:

Parallel Lines: If two lines do not intersect and have the same slope, they are said to be parallel. It is simple to ascertain whether two lines are parallel using the slope-intercept form.

Perpendicular Lines: The slope of a line perpendicular to another line is equal to its negative reciprocal, and two lines are perpendicular if the product of their slopes equals -1.

Distance between a Point and a Line: The distance between a point $(x_1, y_1)$and a line $Ax + By + C = 0$ is given by the formula

d = $|Ax_1 + By_1 + C| / \sqrt{A^2 + B^2}$

Analysing straight lines involves studying their slope, intercepts, intersections, and other geometric properties. These concepts are essential in various areas, such as coordinate geometry, calculus, physics, and engineering, as they provide a foundation for understanding and solving problems involving linear relationships and motion in the plane.

3.6 Slope and Intercept Form

The slope-intercept form is a commonly utilized depiction for the equation of a linear function. It offers important details regarding the line's y-intercept and slope. The equation $y = mx + b$, where m is the line's slope and b is its y-intercept, yields the slope-intercept form of a line.

Slope (m):

The steepness or inclination of a line is determined by its slope. For each unit increase in the x-coordinate, it shows the amount that the y-coordinate changes. The ratio of the vertical change (rise) to the horizontal change (run) between any two locations on the line is known as the slope. Slope (m) can be found mathematically as follows: $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$, where (x_1, y_1) and (x_2, y_2) are any two points on the line.

Properties of the slope:

An upward or rising line is shown by a positive slope (m > 0).

A downhill or falling line is shown by a negative slope (m < 0).

A horizontal line is represented with a slope of zero (m = 0).

A vertical line is represented by an undefinable slope.

Y-Intercept (b):

The point on the y-axis where the line intersects is known as the y-intercept. The constant term "b" in the equation $y = mx + b$ represents the y-intercept in the slope-intercept form. When x = 0, it indicates the y-coordinate of the place on the y-axis where the line crosses.

We can graph a line and find its equation with ease by using the slope and y-intercept. We may plot the y-intercept as a point on the y-axis and use the slope to locate further points on the line given the slope (m) and the y-intercept (b). We may create the line graph by joining these points.

The slope-intercept form comes in handy especially when:

- Calculating a slope of line and y-intercept.
- Plotting the y-intercept and utilizing the slope to discover additional points allows you to graph lines.
- Examining the correlation between variables x and y.
- Resolving issues in physics, engineering, and economics that include linear equations and lines.

Understanding the slope-intercept form allows us to interpret the properties and characteristics of a straight line, including its direction, steepness, and intersection with the y-axis. It provides a versatile tool for representing and analysing linear relationships in analytical plane geometry.

3.7 General equation of first degree

A generic first-degree equation can be used to represent a straight line. Whereas x and y are the variables that represent a point's coordinates on the line, A, B, and C are constants. This is the equation's expression: $Ax + By + C = 0$.

Here's a detailed explanation of the general equation of the first degree:

1. **Coefficients A, B, and C:**

The coefficients A, B, and C in the equation □ □+ □ □+ □ = *0* determine the properties and characteristics of the line. They represent the constants that define the line's position and orientation in the coordinate plane.

2. **A and B coefficients:**

The coefficients A and B in the equation represent the slope of the line. More specifically, the negative ratio A/B gives the slope of the line. The sign of the coefficients determines the direction of the line:

- An upward-sloping line from left to right is represented by a positive value of A
- An downward-sloping line from left to right is represented by a negative value of A.
- An upward-sloping line from bottom to top is represented by a positive number for B, and a downward-sloping line from bottom to top is represented by a negative value for B.

3. **C coefficient**:

The C coefficient in the equation is the constant term. It affects the position of the line relative to the origin (0, 0) and the distance of the line from the origin. The value of C determines the line's intercept with the y-axis. Specifically, C/B gives the y-intercept of the line.

Properties and applications of the general equation of the first degree:

- Intercepts: The equation allows us to determine the x-intercept and y-intercept of the line by setting either x or y to zero and solving for the other variable.
- Slope: The equation allows us to find the slope of the line by comparing the coefficients A and B.
- Parallel and perpendicular lines: Two lines with the same coefficients A and B are parallel, while two lines with negative reciprocal coefficients (-B and A) are perpendicular.
- Distance between a point and a line: Using the formula, one may determine the distance between a (□, □)and the line □ □+ □ □+ □ = *0*.

d = |□□$_1$ + □□$_1$ + □| / $\sqrt{□^2 + □^2}$

The general equation of the first degree is a powerful tool in analytical plane geometry, allowing us to represent and analyse straight lines. By examining the coefficients and understanding their significance, we can gain insights into the line's slope, intercepts, and relationship with other lines.

3.8 Summary

Geometric figures and shapes in a plane are studied using Cartesian coordinates in analytical plane geometry, also referred to as coordinate geometry. It covers ideas like utilizing the distance formula to determine the separation between two places.

The area of a triangle can be calculated using specific formulas based on the coordinates of its vertices. The locus of a point refers to the set of all points that satisfy a given condition. Straight lines can be represented by equations in slope-intercept form or the general equation of the first degree. These concepts and formulas are essential tools for analysing and understanding geometric properties in coordinate geometry.

3.9 Keywords

- **Cartesian Coordinates:** The x- and y-axes, two perpendicular number lines, are used to represent points in a plane in a system known as cartesian coordinates. An ordered pair (x, y) that denotes the horizontal distance from the y-axis and the vertical distance from the x-axis is used to represent each point. In analytical plane geometry, cartesian coordinates offer a foundation for exact position and measurement.
- **Distance Formula:** A mathematical technique called the "distance formula" can be used to determine the separation between two points in a plane. The formula for it is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where x_1, y_1 and $x_2, y_2$are the coordinates of the two points. It is derived from the Pythagorean theorem. The distance formula is a vital tool for the analysis of geometric figures and

shapes because it makes it possible to calculate the straight-line distance between two locations in a coordinate system.

3.10 Self-Assessment Questions

In a Cartesian coordinate system, how far are the points (3, 4) and (-2, 6) apart?

Determine the area in a coordinate plane of a triangle whose vertices are at (1, 2), (4, -1), and (-2, 3).

Determine the locus of points equidistant from the lines x=2 and y= -3 in a Cartesian coordinate system.

In slope-intercept form, write the equation of the straight line that passes through the points (2,5) and (4,9)

For the line with a slope of -2 and a y-intercept of 4, find the general equation of the first degree.

3.11 Case Study

A city's urban planning department is designing a new public park. The design of the park includes complicated geometric shapes and dimensions. As a part of the planning team, you are responsible for precisely determining the coordinates, distances, areas, and equations of different geometric features inside the park using analytical plane geometry techniques. Cartesian coordinates, calculating distances between points, finding triangular areas, identifying point loci, analysing straight lines using slope and intercept form, and comprehending the general equation of the first degree are all covered. The coordinates of three key points in the park: A(2, 4), B(6, 1), and C(8, 5).

1. calculate the distances between these points using the distance formula in Cartesian coordinates.
2. Determine the area of the triangle formed by points A, B, and C using the Shoelace formula or any other applicable method.
3. Discuss how the loci of points can be utilized to identify specific areas within the park that satisfy certain geometric conditions or design requirements.

3.12 References

1. Grewal . B.S., "Elementary Engineering Mathematics", Khanna publications 34th Ed., 2020.
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UNIT : 4

LIMIT & FUNCTION

Learning Objectives:

- To understand Limit of functions
- To understand the differential coefficient,
- To understand differentiation of standard functions

Structure:

4.1 Limit of functions

4.2 Differential coefficient,

4.3 differentiation of standard functions

4.4 Summary

4.5 Keywords

4.6 Self-Assessment Questions

4.7 Case Study

4.8 References

4.1 Limit of functions

In differential calculus, the limit of a function is a fundamental concept to describe behaviour of a function at a specific value. A limit is denoted using notation "lim(x→c) f(x) = L," where f(x) is the function, c is the value the input is approaching, and L is the limit value.

To calculate the limit of a function, consider values of function as the input approaches the given value. Here are some common types of limits and their evaluation methods:

Finite Limits:

Direct Substitution: If substituting the value of c into the function gives a finite value, then that value is the limit.

Example: lim(x→3) (2x - 1) = 2(3) - 1 = 5.

Factorization and Simplification: Factor or simplify the expression and evaluate the limit.

Example: lim(x→2) (x^2 - 4) / (x - 2) = lim(x→2) (x + 2) = 4.

Infinite Limits:

Division by Zero: If the function approaches infinity or negative infinity as the denominator approaches zero, then the limit is infinite.

Example: lim(x→0) 1/x = ∞ (approaches positive infinity).

Vertical Asymptotes: If the function tends to positive or negative infinite then the limit is infinite.

Example: lim(x→∞) (2x + 1) = ∞ (approaches positive infinity).

Limits at Infinity:

Horizontal Asymptotes: If the function approaches a specific value as the input goes to positive or negative infinity, then that value is the limit.

Example: lim(x→∞) (3x^2 + 2x - 1) / (2x^2 - x + 1) = 3/2 (approaches 3/2).

Trigonometric Limits:

Special Limits: Certain trigonometric limits have specific values.

Example: lim(x→0) sin(x) / x = 1 (Squeeze theorem).

These are just a few examples of the methods used to evaluate limits. In practice, there are various algebraic techniques, rules, and theorems that aid in evaluating limits of more complex functions.

4.2 Differential coefficient

In differential calculus, the differential coefficient, measures the changes of function with respect to its input variable. It denoted slope of the tangent line tograph of function at specific point. The first differentiation denoted as f'(x), dy/dx, or df/dx.

The first differentiation of a function f(x) is defined as:

f'(x) = lim(h →0) [f(x + h) - f(x)] / h

This represents immediate rate of change of f(x) at a specific of x. The difference quotient (f(x + h) - f(x)) / h captures mean rate of change over a small interval, taking the limit as h approaches zero gives exact rate of change at exact point.

To obtain the derivative of function, we can use various techniques, including:

Power Rule: For f(x) = ax^n, where a is constant and n is numeral, first differentiation as f'(x) = anx^(n-1).

Ex.: If f(x) = 5x^4 ⇒ f'(x) = 6x.

Sum/Difference rule: For functions that are sums or differences of two or more functions, first differentiation of function is the sum or difference of the derivatives of each distinct function.

Ex.: If f(x) = 8x^5 + 4x^3 - 5x, then f'(x) = 4x^3 + 8x - 5.

Product Rule: For functions that are products of two or more functions, the first differentiation obtains by product rule, in which differentiation of product of 2 functions as u(x) and v(x) is u'(x)v(x) + u(x)v'(x).

Ex.: If f(x) = (2.x^2) (3 x), then f'(x) = (2)(3x) + (2x^2)(3) = 6x + 6x^2.

Quotient Rule: For functions that are ratios of two functions, the first differentiationobtain using the quotient rule, in which differentiation of quotient of 2 functions as u(x) and v(x) is (u'(x) v(x) - u(x) v'(x)) / (v(x))^2.

Ex.: If f(x) = (5x^2) / (2x), then f'(x) = [(2)(3x) - (3x^2)(1)] / (2x)^2 = (6x - 3x^2) / (4x^2).

Chain Rule: For functions that are structures of two or more functions, the derivative can be found using chain rule, which differentiate of f(g(x)) is f'(g(x)) * g'(x).

Example: For $f(x) = \sin(x^2) \Rightarrow f'(x) = \cos(x^2)*2x$

4.3 Differentiation of standard functions

Differentiation use to obtain the differentiation of a function. Here, I will provide the differentiation formulas for some standard functions:

Constant Function: First Differentiation of constant function f(x) = c is zero, when c is constant.

Example: If f(x) = 5, then f'(x) = 0.

Power Rule: let $f(x) = x^n$, where n is numeral, first differentiation is given by $f'(x) = nx^{(n-1)}$.

Example: For $f(x) = x^3 \Rightarrow f'(x) = 3x^2$.

Exponential Function: The differentiation of the exponential $f(x) = e^x$, is equal to the function itself.

Example: If f(x) = e^x, then f'(x) = e^x.

Natural Logarithm Function: The differentiation of the natural logarithm f(x) = ln(x),x is positive numeral, is as f'(x) = 1/x.

Example: If f(x) = ln(x), then f'(x) = 1/x.

Trigonometric Functions:

Sine Function: First differentiation of the sine function f(x) = sin(x) is given by f'(x) = cos(x).

Ex.: If f(x) = sin(x), then f'(x) = cos(x).

Cosine Function: First differentiation of the cosine function f(x) = cos(x) as specified f'(x) = -sin(x).

Example: If f(x) = cos(x), then f'(x) = -sin(x).

Tangent Function: 1st differentiation of the tangent f(x) = tan(x) as specified $f'(x) = Sec^2(x)$, where Sec(x) represents the secant function.

Ex.: If f(x) = tan(x), then $f'(x) = \sec^2(x)$.

Chain Rule

The chain ruleallows us to differentiate compound functions. It is used when we have a function within another function.

Letcompositefunctionf[g(x)], with g(x) function of x and f(u) function of u. First differentiation of f(g(x)) regards to x, using chain rule given as:

$f'(g(x)) * g'(x)$

Here's an example to illustrate the chain rule:

Let the $f(x) = (2x + 1)^3$. We can write it as $f(u) = u^3$, where $u = 2x + 1$. Now, let's differentiate $f(x)$ by use of chain rule:

Step 1: The derivative of $f(u) = u^3 \Rightarrow f'(u) = 3u^2$.

Step 2: Evaluate derivative of inner function $u = 2x + 1$ with respect to x. We get $du/dx = 2$.

Step 3: Apply the chain rule formula: $f'(x) = f'(u) * du/dx$.

Substituting the values we found in steps 1 and 2, we get:

$f'(x) = 3u^2 * 2$

Now, substitute the value of u back into the expression:

$f'(x) = 3(2x + 1)^2 * 2$

Simplifying further, we obtain:

$f'(x) = 6(2x + 1)^2$

So, derivative of $f(x) = (2x + 1)^3$ is $f'(x) = 6(2x + 1)^2$.

The chain rule provides us to differentiate the function that involve compositions of other functions. It is a powerful tool in calculus that helps us handle more complex functions and analyse their rates of change. By understanding and applying the chain rule, we can find the derivatives of functions with nested functions, exponential functions, trigonometric functions, and more.

4.4 Summary

Differential calculus is a field of mathematics that studies derivatives and their applications. It is an effective technique for determining the rate of changes and the attributes of function.

The limit of a function is the first essential idea in differential calculus. The limit of a function indicates the function's behaviour when the input approaches a specific value. It is symbolised by the symbol "lim" and is stated in terms of the function's values when the input approaches the provided value arbitrarily near. Limits enable us to investigate the behaviour of functions at certain points and establish their continuity and differentiability.

The differential coefficient, commonly known as the derivative, is the next crucial notion. The derivative is represented by several notations, including f'(x), dy/dx, and df/dx. It may be calculated using formulae and rules tailored to various sorts of functions.

Finding the derivatives of fundamental functions like polynomials, exponential functions, logarithmic functions, and trigonometric functions is what differentiation of standard functions entails. Each function type has its own set of differentiation rules and formulae.By using the proper rule depending on the function's form, we may get the derivative of any given function.

The chain rule is an important idea in differential calculus since it allows differentiate composite functions. It is used when one function is contained ineach other.

4.5 Keywords

Limit of functions: Represents behaviour of the function as the input approaches a particular value. It is a fundamental concept in differential calculus and is denoted by the symbol "lim". The limit allows us to analyse the behaviour of functions near certain points and determine their continuity and differentiability.

Chain rule: The "chain rule" typically refers to fundamental rule, which is used for evaluation the differentiation of composite functions. However, there isn't a specific "chain rule of differential equations" in the same sense.

In the context of differential equations, the chain rule from calculus is often used when dealing with functions of functions, particularly when solving ordinary or partial differential equations involving compositions of functions.

Chain Rule in Calculus:

If f and g are differentiable functions, then the chain rule states that the derivative of the composite function $f(g(x))$ is given by:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Self-Assessment Questions

1. Evaluate the limit of f(x) = (5x^3 + 7x - 3)/(5x + 2)?
2. Evaluate value of x for which f(x) = x^2 - 3x + 2 has a maximum or minimum.
3. Evaluate 2^{nd}derivative of function f(x) = Sin(2x) + Cos(x).

4. Describe differentiation of f(x) = (2x^3 + 1)^4 by using chain rule.
5. Evaluate the derivative of $f(x) = 5x^2 - 2x^3 + 5x - 6$.

4.7 Case Study

A manufacturing business wants to optimize its production process to increase efficiency and reduce expenses. A mathematician specializing in differential calculus to analyse the company's operations and give ideas for improvement. It is required to discover essential areas, determine maximum and minimum values, and find the best operating conditions for the firm by using differential calculus principles like derivatives and optimization procedures. The cost function associated with the production process is written as:

$C(x) = 5000 + 100x - 2x^2$

where x represents quantity of products produced.

Questions:

1. Calculate derivative of cost function w. r. to x and explain the significance.
2. Determine critical points of cost function and discuss their implications for the production process.

Utilizing differential calculus, find the optimal quantity of products that minimizes the production cost and calculate the corresponding cost value.

4.8 References

1. Grewal . B.S., "Elementary Engineering Mathematics", Khanna publications 34th Ed., 2020.
2. Gupta, S. P and Kapoor V.K, Fundamental of Mathematical Statistics, Sultan Chand and Sons, New Delhi.

UNIT : 5

DERIVATIVES

Learning Objectives:

- To understand differentiation of implicit functions
- To understand the logarithmic differentiation
- To understand parametric differentiation
- To understand successive differentiation

Structure:

5.1 differentiation of implicit functions

5.2 logarithmic differentiation

5.3 parametric differentiation

5.4 successive differentiation

5.5 Summary

5.6 Keywords

5.7 Self-Assessment Questions

5.8 Case Study

5.9 References

5.1 Differentiation of implicit functions

Implicit differentiation is a method used to find derivatives of functions defined implicitly rather than explicitly. It contains differentiation eachsides of an equation with regards to independent variable.

To differentiate an implicit function, we follow these steps:

Step 1: Identify the dependent and independent variables.

Step 2: Differentiate the equation regards to independent variable.

Step 3: Assume dependent variable as independent variable function.

In the differentiation process, we assume y = g(x), where g(x) represents the function of y in terms of x.

Step 4: Apply the chain rule where necessary.

Step 5: Group terms containing the derivative of the dependent variable.

After differentiating both sides of the equation and applying the chain rule, we rearrange the equation to isolate the terms containing the derivative of the dependent variable (dy/dx).

Step 6: Solve the resulting equation for the derivative.

Finally, we solve the resulting equation to find the value of the derivative, dy/dx.

It's important to note that differentiating implicit functions can be more complex than explicit functions since we have to consider the relationship between variables in the equation. It may require algebraic manipulation and application of rules of differentiation to obtain the derivative.

5.2 Logarithmic differentiation

Logarithmic differentiation is utilisedfor differentiating the functions by takingnatural logarithm in eachsides of an equation, then differentiating implicitly, which can simplify the process, especially when dealing with complex functions or products.

Logarithmic Differentiation Formula

The formula for logarithmic differentiation involves the following steps:

1. Take the natural logarithm of both sides of the equation:
   $\ln(f(x)) = \ln(g(x))$
2. Use the properties of logarithms to simplify:
   $\ln(f(x)) = \ln(g(x)) \Rightarrow \ln(f(x)) = \ln(g(x))$
3. Differentiate both sides implicitly with respect to x:
   $\frac{d}{dx}[\ln(f(x))] = \frac{d}{dx}[\ln(g(x))]$
4. Apply the chain rule and other differentiation rules as needed to find the derivatives of $f(x)$ and $g(x)$.
5. Solve for the derivative of the original function $f(x)$ using the derived expression and properties of logarithms if necessary.

5.3 Parametric differentiation

Parametric differentiation involves finding derivatives of parametric equations, where both x and y (or more variables) are expressed separately as functions of a third variable, typically denoted as t. It's useful for analyzing curves and surfaces defined parametrically.

Derivatives Of A Function In Parametric Form

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

To differentiate a parametric equation, we follow these steps:

Step 1: Express x and y in terms of the parameter t.

Givesparametric equations:

$x = f(t)$

$y = g(t)$

Step 2: Obtain derivative of x and y individually with regards t.

Differentiate both x and y with respect to t using the rules of differentiation. Treat x and y as separate functions of t.

Step 3: Compute dy/dx.

To find the derivative dy/dx, divide the derivative of y by the derivative of x:

dy/dx = (dy/dt) / (dx/dt)

Step 4: Simplify the expression.

After computing dy/dx, simplify the resulting expression by combining like terms and applying any algebraic simplifications.

The derivative of a parametric curve with respect to the parameter t may be found via metric differentiation. This is useful when dealing with curves that cannot be described simply as functions of x or y. We may discover how the slope of the curve varies as we go along the parameter t by decomposing x and y independently and then determining dy/dx. This approach is extensively used in physics, engineering, and other areas to analyse and characterise object motion and curve behaviour.

5.4 Successive Differentiation

Successive differentiation, also known as higher-order differentiation, refers to calculate derivatives of function multiple times. It involves taking derivative of function repeatedly, resulting in higher-order derivatives.

To perform successive differentiation, we follow these steps:

Step 1: Start with the original function.

Consider a function f(x) that we want to differentiate.

Step 2: Obtain first differentiation.

Obtain first derivative of f(x) regards x. This gives new f(x), which we denote as f'(x) or dy/dx.

Step 3: Repeat the process for higher-order derivatives.

Continue taking derivatives of the function obtained in the previous step. Each time, we differentiate with respect to x. The 2^{nd} differentiation as f''(x) or d^2y/dx^2, the 3^{rd}derivative as f'''(x) or d^3y/dx^3, and so on.

Step 4: Analyse the higher-order derivatives.

Once we have obtained higher-order derivatives, we can analyse the properties of the function based on these derivatives. For example, we can determine the concavity and inflection points of the function using the second derivative test.

Successive differentiation is useful in a variety of applications, including optimization, curve drawing, and Taylor series expansion. It gives more precise information about a function's behaviour, allowing us to examine its rate of change and curvature at various points. By locating higher-order derivatives, we gain insight into the function's local and global behaviour and can make more accurate predictions about its properties.

5.5 Summary

Differentiation is a fundamental topic in calculus that involves determining a function's rate of change. While traditional differentiation techniques are usually employed for explicit functions, there are a number of specialised approaches available for dealing with more complicated circumstances. Differentiation of implicit functions, logarithmic differentiation, parametric differentiation, and sequential differentiation are four such approaches.

When a function is not explicitly specified in terms of a dependent variable, implicit function differentiation is utilised. It is instead presented as an equation that includes both the dependent and independent variables. Each sidesin equation, differentiating regard to independent variable by using chain rule to obtain derivative of dependent variable are both steps in the procedure.

When working with functions including products, quotients, or powers, logarithmic differentiation is a technique used to simplify differentiation. We may utilise the principles of logarithms to change an equation into a form that is easier to distinguish by convert in logarithm ofeachsides. We can reconstruct original function by exponentiating both sides after getting the derivative.

When a function is defined parametrically, meaning it is stated in terms of one or more parameters, parametric differentiation is used. We may obtain the differentiation of dependent variables in regard to independent variable by differentiating each parameter with respect to the independent variable and applying the chain rule.

The process of obtaining higher-order derivatives of a function is referred to as successive differentiation. It entails taking numerous derivatives of a function to generate second, third, and higher-order derivatives. Higher-order derivatives reveal more about the function's behaviour and features, such as concavity and inflection points.

These specialised procedures broaden our differentiation scope, allowing us to handle more complicated tasks and circumstances. They are useful for addressing issues involving implicit equations, simplifying difficult functions, dealing with parametric equations, and analysing higher-order function behaviour. We may obtain deeper insights into the behaviour of functions and solve a larger range of mathematical issues by learning and implementing these strategies.

5.6 Keywords

- Implicit Differentiation
- Logarithmic Differentiation

Self-Assessment Questions

1. In $x^2 + y^2 = 25$, find dy/dx using implicit differentiation.
2. Given the equation $y = e^{(2x^2)}$, use logarithmic differentiation to find dy/dx.
3. Consider the parametric equations $x = 2t^2$ and $y = 3t + 1$. Find dy/dx using parametric differentiation.
4. Obtain second differentiation of $f(x) = 7x^5 - 3x^2 + 5x^3 - 9x + 2$ using successive differentiation.
5. Given the equation $x^2 + y^3 - 4xy = 6$, use implicit differentiation to find dy/dx.

5.8 Case Study

X works as a manager for a company that manufactures cylindrical cans. The company seeks to maximize can volume while minimizing the quantity of material needed in manufacture. The cans have a constant height of 10 centimeters, and the radius of the can be adjusted.

Questions:

1. Using the formula for the volume of the cylindrical can, V = πr^2h, derive an expression for the radius that maximizes the volume while keeping the height fixed at 10 centimeters. Apply implicit differentiationto obtain critical points and check it is maximum or minimum.
2. Suppose, manufacturing company decides to change the height of the cylindrical cans from 10 centimeters to 12 centimeters. Using the logarithmic differentiation technique, calculate ratio changes involume of the cans as outcome of this height adjustment.
3. A parametric equation x = rcos(t) &y = rsin(t) for curve of cylindrical cans, obtain an expression for curvature in standing of r, t, & their derivatives. Evaluate the curvature at a specific point on the curve to measure rate of change of curvature relating to the radius.

5.9 References

1. Grewal . B.S., "Elementary Engineering Mathematics", Khanna publications 34th Ed., 2020.
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UNIT : 6

INTEGRATION

Learning Objectives:

- To understand Integration as inverse of differentiation
- To understand integration by parts, by partial and by substitution
- To understand formal evaluation of definite integrals

Structure:

6.1 Integration as inverse of differentiation

6.2 Indefinite integrals of standard forms

6.3 Integration by parts

6.4 Integration by partial and by substitution

6.5 Formal evaluation of definite integrals

6.6 Summary

6.7 Keywords

6.8 Self-Assessment Questions

6.9 Case Study

6.10 References

6.1 Integration as inverse of differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.
Let us consider the following examples:

We know that $$\frac{d}{dx}(\sin x) = \cos x \qquad \text{... (1)}$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \qquad \text{... (2)}$$

and $$\frac{d}{dx}(e^x) = e^x \qquad \text{... (3)}$$

We observe that in (1), the function $\cos x$ is the derived function of $\sin x$. We say that $\sin x$ is an anti derivative (or an integral) of $\cos x$. Similarly, in (2) and (3), $\frac{x^3}{3}$ and e^x are the anti derivatives (or integrals) of x^2 and e^x, respectively. Again, we note that for any real number C, treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + C) = \cos x, \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2 \text{ and } \frac{d}{dx}(e^x + C) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as ***arbitrary constant***. In fact, C is the ***parameter*** by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function F such that $\frac{d}{dx}F(x) = f(x)$, $\forall x \in$ I (interval), then for any arbitrary real number C, (also called *constant of integration*)

$$\frac{d}{dx}\left[F(x) + C\right] = f(x), \; x \in I$$

Thus, $\{F + C, C \in \mathbf{R}\}$ denotes a family of anti derivatives of f.

Remark Functions with same derivatives differ by a constant. To show this, let g and h be two functions having the same derivatives on an interval I.

Consider the function $f = g - h$ defined by $f(x) = g(x) - h(x)$, $\forall x \in I$

Then $\frac{df}{dx} = f' = g' - h'$ giving $f'(x) = g'(x) - h'(x)$ $\forall x \in I$

or $f'(x) = 0$, $\forall x \in I$ by hypothesis,

i.e., the rate of change of f with respect to x is zero on I and hence f is constant.

In view of the above remark, it is justified to infer that the family $\{F + C, C \in \mathbf{R}\}$ provides all possible anti derivatives of f.

We introduce a new symbol, namely, $\int f(x)\,dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x.

Symbolically, we write $\int f(x)\,dx = F(x) + C$.

Notation Given that $\frac{dy}{dx} = f(x)$, we write $y = \int f(x)\,dx$.

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
Particularly, we note that $\frac{d}{dx}(x) = 1$;	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\,dx = \sin x + C$
(iii) $\frac{d}{dx}(-\cos x) = \sin x$;	$\int \sin x\,dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x\,dx = \tan x + C$
(v) $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$;	$\int \text{cosec}^2 x\,dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x\,dx = \sec x + C$
(vii) $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$;	$\int \text{cosec}\, x \cot x\,dx = -\text{cosec}\, x + C$
(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
(ix) $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$; $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\frac{d}{dx}\left(-\cot^{-1} x\right) = \frac{1}{1+x^2}$; $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x dx = e^x + C$

(xv) $\frac{d}{dx}(\log|x|) = \frac{1}{x}$; $\int \frac{1}{x}\, dx = \log|x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$; $\int a^x dx = \frac{a^x}{\log a} + C$

6.2 Indefinite integrals of standard forms

In this sub section, we shall derive some properties of indefinite integrals.

(I) The process of differentiation and integration are inverses of each other in the sense of the following results :

$$\frac{d}{dx}\int f(x)\, dx = f(x)$$

and $\int f'(x)\, dx = f(x) + C$, where C is any arbitrary constant.

Proof Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx}\mathrm{F}(x) = f(x)$$

Then $$\int f(x)\,dx = \mathrm{F}(x) + \mathrm{C}$$

Therefore $$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}(\mathrm{F}(x)+\mathrm{C})$$

$$= \frac{d}{dx}\mathrm{F}(x) = f(x)$$

Similarly, we note that

$$f'(x) = \frac{d}{dx}f(x)$$

and hence $$\int f'(x)\,dx = f(x) + \mathrm{C}$$

where C is arbitrary constant called constant of integration.

(II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

Proof Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$$

or $$\frac{d}{dx}\left[\int f(x)\,dx - \int g(x)\,dx\right] = 0$$

Hence $\int f(x)\,dx - \int g(x)\,dx =$ C, where C is any real number (Why?)

or $$\int f(x)\,dx = \int g(x)\,dx + \mathrm{C}$$

So the families of curves $\left\{\int f(x)\,dx + \mathrm{C}_1, \mathrm{C}_1 \in \mathbf{R}\right\}$

and $\left\{\int g(x)\,dx + \mathrm{C}_2, \mathrm{C}_2 \in \mathbf{R}\right\}$ are identical.

Hence, in this sense, $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

(III) $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

Proof By Property (I), we have

$$\frac{d}{dx}\left[\int [f(x) + g(x)]\, dx\right] = f(x) + g(x) \quad \text{... (1)}$$

On the otherhand, we find that

$$\frac{d}{dx}\left[\int f(x)\, dx + \int g(x)\, dx\right] = \frac{d}{dx}\int f(x)\, dx + \frac{d}{dx}\int g(x)\, dx$$

$$= f(x) + g(x) \quad \text{... (2)}$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int \left(f(x) + g(x)\right) dx = \int f(x)\, dx + \int g(x)\, dx .$$

(IV) For any real number k, $\int k\, f(x)\, dx = k \int f(x)\, dx$

Proof By the Property (I), $\frac{d}{dx}\int k\, f(x)\, dx = k\, f(x)$.

Also $\quad \frac{d}{dx}\left[k \int f(x)\, dx\right] = k\frac{d}{dx}\int f(x)\, dx = k\, f(x)$

Therefore, using the Property (II), we have $\int k\, f(x)\, dx = k \int f(x)\, dx$.

(V) Properties (III) and (IV) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int [k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]\, dx$$

$$= k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx + ... + k_n \int f_n(x)\, dx .$$

Example 1 Write an anti derivative for each of the following functions using the method of inspection:

(i) $\cos 2x$ (ii) $3x^2 + 4x^3$ (iii) $\frac{1}{x}, x \neq 0$

Solution

(i) We look for a function whose derivative is $\cos 2x$. Recall that

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

or $$\cos 2x = \frac{1}{2}\frac{d}{dx} (\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

Therefore, an anti derivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$.

(ii) We look for a function whose derivative is $3x^2 + 4x^3$. Note that

$$\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3.$$

Therefore, an anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$.

(iii) We know that

$$\frac{d}{dx}(\log x) = \frac{1}{x}, x > 0 \text{ and } \frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}, x < 0$$

Combining above, we get $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}, x \neq 0$

Therefore, $\int \frac{1}{x}\, dx = \log|x|$ is one of the anti derivatives of $\frac{1}{x}$.

Example 2 Find the following integrals:

(i) $\int \frac{x^3 - 1}{x^2} dx$ (ii) $\int (x^{\frac{2}{3}} + 1)\, dx$ (iii) $\int (x^{\frac{3}{2}} + 2 e^x - \frac{1}{x})\, dx$

Solution

(i) We have

$$\int \frac{x^3 - 1}{x^2}\, dx = \int x\, dx - \int x^{-2}\, dx \quad \text{(by Property V)}$$

$$= \left(\frac{x^{1+1}}{1+1} + C_1\right) - \left(\frac{x^{-2+1}}{-2+1} + C_2\right); \quad C_1, C_2 \text{ are constants of integration}$$

$$= \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2$$

$$= \frac{x^2}{2} + \frac{1}{x} + C, \text{ where } C = C_1 - C_2 \text{ is another constant of integration.}$$

Note From now onwards, we shall write only one constant of integration in the final answer.

(ii) We have

$$\int (x^{\frac{2}{3}} + 1)\, dx = \int x^{\frac{2}{3}}\, dx + \int dx$$

$$= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + x + C = \frac{3}{5} x^{\frac{5}{3}} + x + C$$

(iii) We have $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\, dx = \int x^{\frac{3}{2}}\, dx + \int 2e^x\, dx - \int \frac{1}{x}\, dx$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2e^x - \log|x| + C$$

$$= \frac{2}{5} x^{\frac{5}{2}} + 2e^x - \log|x| + C$$

6.3 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u\frac{dv}{dx}\, dx + \int v\frac{du}{dx}\, dx$$

or $$\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx \qquad \text{... (1)}$$

Let $u = f(x)$ and $\frac{dv}{dx} = g(x)$. Then

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x)\, dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [\int g(x)\, dx]\, f'(x)\, dx$$

i.e.,

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [f'(x) \int g(x)\, dx]\, dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example Find $\int x \cos x\, dx$

Solution Put $f(x) = x$ (first function) and $g(x) = \cos x$ (second function).
Then, integration by parts gives

$$\int x \cos x\, dx = x \int \cos x\, dx - \int [\frac{d}{dx}(x) \int \cos x\, dx]\, dx$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + C$$

Suppose, we take $f(x) = \cos x$ and $g(x) = x$. Then

$$\int x \cos x\, dx = \cos x \int x\, dx - \int [\frac{d}{dx}(\cos x) \int x\, dx]\, dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x \frac{x^2}{2} dx$$

Thus, it shows that the integral $\int x \cos x\, dx$ is reduced to the comparatively more complicated integral having more power of x. Therefore, the proper choice of the first function and the second function is significant.

Example Find $\int \log x\, dx$

Solution To start with, we are unable to guess a function whose derivative is $\log x$. We take $\log x$ as the first function and the constant function 1 as the second function. Then, the integral of the second function is x.

Hence,

$$\int (\log x.1)\, dx = \log x \int 1\, dx - \int [\frac{d}{dx}(\log x) \int 1\, dx]\, dx$$

$$= (\log x) \cdot x - \int \frac{1}{x} x\, dx = x \log x - x + C.$$

Example Find $\int x e^x dx$

Solution Take first function as x and second function as e^x. The integral of the second function is e^x.

Therefore,

$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = xe^x - e^x + C.$$

6.4 Integration by partial and by substitution

In this section, we consider the method of integration by substitution.

The given integral $\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider $$I = \int f(x)\,dx$$

Put $x = g(t)$ so that $\frac{dx}{dt} = g'(t)$.

We write $$dx = g'(t)\,dt$$

Thus $$I = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

Example Integrate the following functions w.r.t. x:

(i) $\sin mx$ (ii) $2x \sin(x^2 + 1)$

(iii) $\frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$ (iv) $\frac{\sin(\tan^{-1} x)}{1 + x^2}$

Solution

(i) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $m\,dx = dt$.

Therefore, $$\int \sin mx\,dx = \frac{1}{m}\int \sin t\,dt = -\frac{1}{m}\cos t + C = -\frac{1}{m}\cos mx + C$$

(ii) Derivative of $x^2 + 1$ is $2x$. Thus, we use the substitution $x^2 + 1 = t$ so that $2x\, dx = dt$.

Therefore, $\int 2x \sin (x^2 + 1)\, dx = \int \sin t\, dt = -\cos t + C = -\cos (x^2 + 1) + C$

(iii) Derivative of $\sqrt{x}$ is $\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Thus, we use the substitution

$\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}} dx = dt$ giving $dx = 2t\, dt$.

Thus, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \int \frac{2t \tan^4 t \sec^2 t\, dt}{t} = 2\int \tan^4 t\ \sec^2 t\, dt$$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t\, dt = du$

Therefore, $$2\int \tan^4 t \sec^2 t\, dt = 2\int u^4\, du = 2\frac{u^5}{5} + C$$

$$= \frac{2}{5} \tan^5 t + C \ \text{(since } u = \tan t)$$

$$= \frac{2}{5} \tan^5 \sqrt{x} + C \ \text{(since } t = \sqrt{x})$$

Hence, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \frac{2}{5} \tan^5 \sqrt{x} + C$$

Alternatively, make the substitution $\tan \sqrt{x} = t$

(iv) Derivative of $\tan^{-1} x = \frac{1}{1+x^2}$. Thus, we use the substitution

$$\tan^{-1} x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore, $$\int \frac{\sin (\tan^{-1} x)}{1+x^2} dx = \int \sin t\, dt = -\cos t + C = -\cos (\tan^{-1} x) + C$$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\boldsymbol{\int \tan x\, dx = \log|\sec x| + C}$

We have

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

Put $\cos x = t$ so that $\sin x\, dx = -dt$

Then $\int \tan x\, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$

or $\int \tan x\, dx = \log|\sec x| + C$

(ii) $\int \cot x\, dx = \log|\sin x| + C$

We have $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

Put $\sin x = t$ so that $\cos x\, dx = dt$

Then $\int \cot x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x\, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

Put $\sec x + \tan x = t$ so that $\sec x\,(\tan x + \sec x)\, dx = dt$

Therefore, $\int \sec x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)}\, dx$

Put $\operatorname{cosec} x + \cot x = t$ so that $-\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\, dx = dt$

So $\int \operatorname{cosec} x\, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C$

$$= -\log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x}\right| + C$$

$$= \log|\operatorname{cosec} x - \cot x| + C$$

Integration by the Partial Fractions Method

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(2) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

(3) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(4) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(5) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

We now prove the above results:

(1) We have $\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Therefore, $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$

$$= \frac{1}{2a}\left[\log|(x-a)| - \log|(x+a)|\right] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a+x)(a-x)}\right] = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

Therefore, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$

$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Note The technique used in (1) will be explained in Section 7.5.

(3) Put $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

Therefore, $\int \frac{dx}{x^2 + a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) Let $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C, \text{ where } C = C_1 - \log|a|$$

(5) Let $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$

$$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C$$

(6) Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$

$$= \int \sec\theta\, d\theta = \log|(\sec\theta + \tan\theta)| + C_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 + a^2}\right| - \log|a| + C_1$$

$$= \log\left|x + \sqrt{x^2 + a^2}\right| + C, \text{ where } C = C_1 - \log|a|$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,
	where $x^2 + bx + c$ cannot be factorised further	

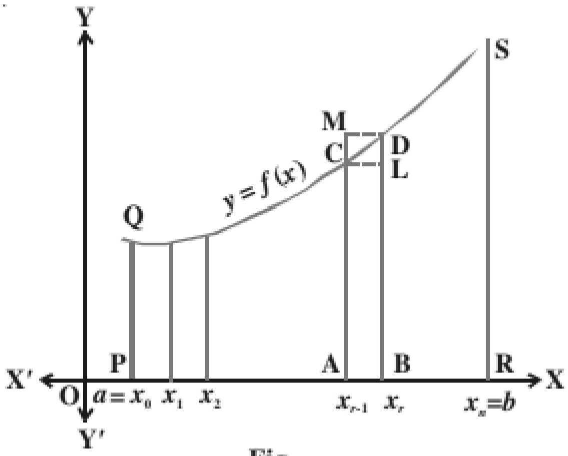
6.5 Formal evaluation of definite integral

A definite integral is a specific type of integral that involves the evaluation of a function over a given interval. It is characterised by having both the upper and lower limits of integration specified. The interval, denoted as [a, b], represents the range over which the function is integrated. The definite integral is also referred to as the Riemann integral, named after the mathematician Bernhard Riemann who made significant contributions to the theory of integration.

The definite integral has numerous applications across various fields, including physics, engineering, economics, and probability. It is used to calculate areas, determine the total accumulated quantity, find the average value of a function, and solve various real-life problems. By computing the definite integral, we gain precise information about the behaviour and properties of functions within a specific interval, enabling us to make accurate mathematical predictions and analyses.

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$



Fig

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,...., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \frac{b-a}{h}$. We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig , we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \ldots (2)$$

and $$S_n = h\,[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \qquad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \qquad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+(n-1)h]$$

or $$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)h] \qquad \ldots (6)$$

where $$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum.*

Example Find $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Solution By definition

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)h],$$

where, $h = \dfrac{b-a}{n}$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

Therefore,

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-terms}}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\frac{(n-1)\;(2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}[1+\frac{2}{3}(1-\frac{1}{n})\;(2-\frac{1}{n})] = 2\;[1+\frac{4}{3}] = \frac{14}{3}$$

Example Evaluate $\int_0^2 e^x \, dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x \, dx = (2-0) \lim_{n \to \infty} \frac{1}{n} \left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}} \right]$$

Using the sum to n terms of a G.P., where $a = 1$, $r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x \, dx = 2 \lim_{n \to \infty} \frac{1}{n} \left[\frac{e^{\frac{2n}{n}} - 1}{e^{\frac{2}{n}} - 1} \right] = 2 \lim_{n \to \infty} \frac{1}{n} \left[\frac{e^2 - 1}{e^{\frac{2}{n}} - 1} \right]$$

$$= \frac{2(e^2-1)}{\lim_{n \to \infty} \left[\frac{e^{\frac{2}{n}} - 1}{\frac{2}{n}} \right] \cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h \to 0} \frac{(e^h - 1)}{h} = 1\right]$$

EXERCISE

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x \, dx$
2. $\int_0^5 (x+1) \, dx$
3. $\int_2^3 x^2 \, dx$
4. $\frac{x}{(x-1)(x-2)(x-3)}$
5. $\frac{2x}{x^2+3x+2}$
6. $\frac{1-x^2}{x(1-2x)}$
7. $\sqrt{1+3x-x^2}$
8. $\sqrt{x^2+3x}$
9. $\sqrt{1+\frac{x^2}{9}}$
10. $\frac{1}{\sqrt{x^2+2x+2}}$
11. $\frac{1}{9x^2+6x+5}$
12. $\frac{1}{\sqrt{7-6x-x^2}}$
13. $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
14. $\frac{\cos x - \sin x}{1+\sin 2x}$
15. $\tan^3 2x \sec 2x$
16. $\tan^4 x$
17. $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
18. $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
19. $\frac{1}{\sin x \cos^3 x}$
20. $\frac{\cos 2x}{(\cos x + \sin x)^2}$
21. $\sin^{-1}(\cos x)$

6.6 Summary

Integral Calculus addresses the process of learning integration, which is differentiation inverted. It is concerned with determining anti-derivatives and assessing definite integrals. Fundamental concepts in Integral Calculus include integration as a reverse of differentiation, indefinite integrals of standard forms, integration by parts, partial fractions, substitution, and the formal evaluation of definite integrals.

Ruling the anti-derivative of a function is what integration as the inverse of differentiation entails. By integrating, we may reconstruct the original function from its derivative. The indefinite integral symbol represents this operation, and the anti-derivative is derived by adding an integration constant.

The integration of common functions is referred to as indefinite integrals of standard forms. For integrating standard forms, well-known formulas exist, such as power, exponential, logarithmic, trigonometric, and inverse trigonometric functions. These formulae give a methodical approach to locating anti-derivatives.

Integration by parts is a technique for combining the results of two functions. It is depended on the multiple rule of differentiation and entails choosing one function as and the other as then using the integration by parts procedure to produce the integral.

Integration by partial fractions is a process for breaking down a rational function into smaller fractions. When integrating rational functions with denominators that may be factored into linear or quadratic terms, this approach comes in handy.

Integration by substitution, also identified as variable change, is the progression of replacing a new variable in an integral to simplify its form. This approach allows the integral to be transformed into a known form or a complex integral to be reduced to a simpler one.

Calculating the value of a definite integral over a specified interval is what formal evaluation of definite integrals entails. This is accomplished by determining the function's anti-derivative and evaluating it at the upper limit and lower limit of integration, then subtracting the two results to produce the numerical value of the definite integral.

6.7 Keywords

- **Anti-derivative:** The anti-derivative, also recognized as the indefinite integral, is the reverse procedure of differentiation. It represents the original function from which the derivative is obtained.

- **Integration Techniques:** Integration techniques refer to various methods used to evaluate integrals, for example integration by parts, by partial fractions, and by replacement. These techniques provide tools for solving a wide range of integration problems.

Self-Assessment Questions

1. Evaluate the indefinite integral of the function $f(x) = 3zx^2 + 2x - 5$.
2. Apply integration by parts to evaluate the definite integral $\int xe^x dx$ over the interval [0, 1].
3. Find the indefinite integral of the function $g(x) = (x^2 + 1) / x^3$.
4. Use the method of partial fractions to evaluate the definite integral $\int (3x + 2) / (x^2 - 4)\, dx$ over the interval [-2, 2].
5. Apply the substitution method to evaluate the definite integral $\int (2x + 1) / \sqrt{(x^2 + 3)}\, dx$ over the interval [-1, 2].

6.9 Case Study

X is a civil engineer entrusted with creating a river bridge. To ensure the bridge's safety, X must calculate the overall area of a portion of the riverbed. X has collected data points reflecting water depth at regular intervals along the river. The area under the curve created by these data points is the task of X.

Question:

1. Let's assume you have obtained the following data points representing the depth of the water in meters at 1-meter intervals along the river: $(0, 2), (1, 3), (2, 5), (3, 4), (4, 6), (5, 7), (6, 8)$. How can you use integration to calculate the total area under the curve formed by these data points?

2. Using the data points mentioned above, apply the definite integral to calculate the total area under the curve from $x = 0$to $\square = 6$. Show your step-by-step calculation process.
3. In the scenario of designing the bridge, why is it necessary to compute the area covering by the curve? How does this calculation help in ensuring the safety of the bridge?

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UNIT : 7

FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure:

7.1 Differential equations

7.2 Definition and construction of ordinary differential equations

7.3 Eq. of first order and first degree

7.4 Variable separable differentiation

7.5 Outline

7.6 Keywords

7.7 Self-Assessment Questions

7.8 Case Study

7.9 References

7.1 Differential equations

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $xdx + ydy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (vi) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Order of Dfferential equations

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation after the equation is free from negative and fractional powers of the derivatives. For example,

	Differential Equation	Order	Degree
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

First Order Differential equations

A **First Order Linear Differential Equation** is a first order differential equation which can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x), Q(x)$ are continuous functions of x on a given interval.

The above form of the equation is called the **Standard Form** of the equation.

Example Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y.$$

To solve an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we multiply by a function of of x called an **Integrating Factor**. This function is

$$\boxed{I(x) = e^{\int P(x)dx}.}$$

(we use a particular antiderivative of $P(x)$ in this equation.)
$I(x)$ has the property that

$$\frac{dI(x)}{dx} = P(x)I(x).$$

Multiplying across by $I(x)$, we get an equation of the form

$$I(x)y' + I(x)P(x)y = I(x)Q(x).$$

The left hand side of the above equation is the derivative of the product $I(x)y$. Therefore we can rewrite our equation as

$$\frac{d[I(x)y]}{dx} = I(x)Q(x).$$

Integrating both sides with respect to x, we get

$$\int \frac{d[I(x)y]}{dx}dx = \int I(x)Q(x)dx$$

or

$$I(x)y = \int I(x)Q(x)dx + C$$

giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

(we amalgamate constants in this equation.)

Second-Order Differential equations

A **Second Order Linear Differential Equation** is a second order differential equation which can be put in the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where $P(x), Q(x), R(x)$ and $G(x)$ are continuous functions of x on a given interval.

If $G = 0$ then the equation is called **homogeneous**. Otherwise is called **nonhomogeneous**. In this lecture, we will solve homogeneous second order linear equations, in the next lecture, we will cover nonhomogeneous second order linear equations. In general is very difficult to solve second order linear equations, general ones will be solved in a differential equations class. Here we will solve homogeneous second order linear equations with constant coefficients, i.e. equations of the type:

$$ay'' + by' + cy = 0,$$

where a, b and c are constants and $a \neq 0$.

Example $y'' + 2y' - 8y + 0$ is a homogeneous second order linear equation with constant coefficients.

7.2 Definition and formation of ordinary differential equations

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad \cos^2 x \frac{d^3y}{dx^3} + x^3 \frac{dy}{dx} + y = 0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y \frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable y and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

Example of a simple ordinary differential equation:

$$\frac{dy}{dx} = 2x$$

This equation represents the rate of change of y with respect to x being equal to $2x$. It's a first-order linear ODE, where the solution would be a function $y(x)$ that satisfies this equation.

7.3 Eq. of first order and first degree

The general form of first order ,first degree differential equation is $\frac{dy}{dx} = f(x,y)$ or [Mdx + Ndy =0 Where M and N are functions of x and y]. There is no general method to solve any first order differential equation The equation which belong to one of the following types can be easily solved.

In general the first order differential equation can be classified as:

(1). Variable separable type

(2). (a) Homogeneous equation and

(b)Non-Homogeneous equations which to exact equations.

(3) (a) exact equations and

(b) equations reducible to exact equations.

4) (a) Linear equation &

(b) Bernoulli's equation.

7.4 Variable separable

If the differential equation $\frac{dy}{dx}$=f(x,y) can be expressed of the form $\frac{dy}{dx}=\frac{f(x)}{g(y)}$ or f(x) dx –g(y)dy =0 where f and g are continuous functions of a single variable, then it is said to be of the form variable separable.

General solution of variable separable is $\int f(x)dx - \int g(y)dy$= c

Where c is any arbitrary constant.

PROBLEMS:

1) tan y $\frac{dy}{dx}$ = sin(x+y) + sin(x-y).

Sol: Given that sin(x+y) + sin(x-y) = tan y $\frac{dy}{dx}$

⇨ 2sinx.cosx = tan y $\frac{dy}{dx}$ [**Note**: sinC+sinD =2sin($\frac{C+D}{2}$).cos($\frac{C-D}{2}$)]

⇨ 2sinx = tany secy $\frac{dy}{dx}$

General solution is $2\int \sin x \, dx = \int secy\,.tany\,.dy$

=> -2cosx = secy +c

=> sec y + 2 cos x +c =0 .//

2) Solve $(x^2+1).\frac{dy}{dx}+(y^2+1)=0$, **y(0) =1.**

Sol: Given $(x^2+1).\frac{dy}{dx}+(y^2+1)=0$

$$\Rightarrow \quad \frac{dx}{x^2+1}+\frac{dy}{y^2+1}=0$$

On Integrations

$$\Rightarrow \int\frac{1}{(1+x^2)}dx+\int\frac{1}{(1+y^2)}dy=0$$

$\Rightarrow \tan^{-1}x+\tan^{-1}y=c$ ---------------(1)

Given y(0)=1 => At x=0 ,y=1 ---------(2)

(2) in (1) $\Rightarrow \tan^{-1}0+\tan^{-1}1=c.$

$\Rightarrow 0+\frac{\pi}{4}=c$

$\Rightarrow c=\frac{\pi}{4}.$

Hence the required solution is $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$

Exact Differential Equations:

Def: Let M(x,y)dx +N(x,y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x,y . Then the equation Mdx + Ndy =0 is said to be an exact Differential equation if ∃ a function f ∋.

$$d[f(x,y)]=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Condition for Exactness: If M(x,y) & N (x,y) are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation Mdx+ Ndy =0 is to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Hence solution of the exact equation M(x,y)dx +N(x,y) dy =0. Is

$$\int Mdx + \int Ndy = c.$$

(y constant) (terms free from x).

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PROBLEMS

1) Solve $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Sol: Hence M = $1+e^{\frac{x}{y}}$ & N = $e^{\frac{x}{y}}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \& \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + (1-\frac{x}{y})e^{\frac{x}{y}}(\frac{1}{y})$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \& \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ equation is exact

General solution is

$$\int M dx \quad + \int N dy \ = c.$$

(y constant) (terms free from x)

$$\int(1+e^{\frac{x}{y}})\,dx \quad + \int 0\,dy \ = c.$$

$$=> x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$=> x + y\,e^{\frac{x}{y}} = C$$

2. Solve (e^y+1) .cosx dx + $e^y \sin x\, dy$ =0.

Ans: (e^y+1) . sinx =c $\qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^x \cos x$

3. Solve (r+sinθ $-$casθ) dr + r (sin$\theta + cos\theta$) d$\theta = 0$.

Ans: $r^2+2r(\sin\theta-\cos\theta)=2c$

$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$ = sin$\theta + cos\theta$.

4. Solve $[y(1+\frac{1}{x})+\cos y]\,dx+[x+\log x-x\sin y]dy=0$.

Sol: hence $M=y(1+\frac{1}{x})+\cos y$, $N=x+\log x-x\sin y$.

$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$ $\quad\quad$ $\frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the equation is exact

$$\text{General sol } \int M dx + \int N dy = c.$$

(y constant) (terms free from x)

$$\int[y+\frac{y}{x}+\cos y]dx + \int o\,.dy = c.$$

$\Rightarrow y(x+\log x)+x\cos y=c.$

5. Solve $y\sin 2x\,dx-(y^2+\cos x)\,.dy=0$.

6. Solve $(\cos x-x\cos y)dy-(\sin y+(y\sin x))dx=0$

Sol: $N=\cos x-x\cos y$ & $M=-\sin y-y\sin x$

$\frac{\partial N}{\partial x}=-\sin x-\cos y$ $\quad\quad$ $\frac{\partial M}{\partial y}=-\cos y-\sin x$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow$ the equation is exact.

$$\text{General sol } \int M dx + \int N dy = c.$$

(y constant) (terms free from x)

$$\Rightarrow\int(-\sin y-y\sin x).dx + \int o\,.dy = c$$

$\Rightarrow -x\sin y+y\cos x=c$

$\Rightarrow y\cos x-x\sin y=c.$

7. Solve $(\sin x\,.\sin y-x\,e^{y})\,dy=(e^{y}+\cos x-\cos y)\,dx$

Ans: $xe^{y}+\sin x.\cos y=c.$

8. Solve $(x^2+y^2-a^2)\, x\, dx + (x^2-y^2-b^2) . y .dy = 0$

Ans: $x^4+2x^2y^2-2a^2x^2-2b^2y^2 = c$.

7.5 Summary

Differential equations are mathematical formulas that use derivatives to explain the relationship between a function and its derivatives. Ordinary differential equations (ODEs) are used to solve functions with a single variable. ODEs are created by expressing the derivatives of an unknown function in terms of the independent variable and the function itself. ODEs of first order and first degree involve the first derivative of an unknown function. Variable separable is a method for solving ODEs in which the variables may be separated on either side of the problem. It entails decoupling the variables, integrating both sides, and arriving at a general solution.

7.6 Keywords

1. **Ordinary$d$ifferential equations ODEs):** equations in mathematics that deal with derivatives and explain how a function and its derivatives are related.
2. **Variable separable:** A technique used to solve ODEs where the variables can be separated on either side of the equation.

7.7 Self-Assessment Questions

(a) Order of differential equation: Order of the highest derivative occurring in the differential equation

(b) Degree of differential equation: Degree of the highest order derivative when differential coefficients are free from radicals and fractions.

(c) General equation : $\frac{dy}{dx} = f(x) \Rightarrow \quad y = \int f(x)dx + c$

(d) $\frac{dy}{dx} = f(ax+by+c)$, then put $ax + by + c = v$

(e) If $\frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1}dy = f(x)dx$ then $\int (g(y))^{-1}dy = \int f(x)dx$

7.8 Case Study

A is a biologist who is researching the population dynamics of a certain species in an ecosystem. The species' population is influenced by a variety of factors, including birth rate, mortality rate, and accessible resources. Your objective is to use differential equations to simulate population increase and analyse population behaviour over time.

Question

1. Given the birth rate of the species as 0.05 individuals per day and the mortality rate as 0.03 individuals per day, calculate the net population growth rate per day.
2. Suppose the accessible resources for the species decrease over time, causing a decline in the birth rate from 0.05 to 0.03 individuals per day. If the mortality rate remains constant at 0.03 individuals per day, calculate the new net population growth rate and the equilibrium population size assuming the birth and mortality rates remain constant.

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UNIT : 8

HOMOGENOUS AND NON-HOMOGENOUS DIFFERENTIAL EQUATIONS

Learning Objectives:

- To understand Homogeneous equations
- To understand nonhomogeneous equations
- To understand linear equations
- To understand differential equations

Structure:

8.1 Homogeneous equations

A linear differential equation of the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

is called a homogeneous linear differential equation because the right-hand side is zero. If the right-hand side were a non-zero function of xxx, it would be a non-homogeneous linear differential equation.

Example

$$y'' + p(x)y' + q(x)y = 0$$

8.2 Non-homogeneous equations

A linear non-homogeneous differential equation has the general form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

Where g(x) is a non-zero function.

Example:

$$y'' + p(x)y' + q(x)y = g(x)$$

SOLVED EXAMPLES

Example 1. Solve $(D^2 - D'^2 + D - D')z = 0$.

Solution. The given partial differential equation can be written as

$$(D - D')(D + D' + 1)z = 0 \qquad \ldots(1)$$

Here, we see that equation (1) is a **reducible non-homogeneous linear partial differential equation**.

The part of complementary function C.F. corresponding to the factor $(D - D')$ is $\phi_1(y + x)$.

Again, the part of complementary function C.F. corresponding to the factor $(D + D' + 1)$ is $e^{-x}\phi_2(y - x)$.

Therefore, the general solution of (1) is given by

$$z = \phi_1(y + x) + e^{-x}\phi_2(y - x) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Example 2. Solve $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0$.

Solution. The given partial differential equation can be written as

$$[(D + aD')(D - aD') + 2ab(D + aD')]z = 0$$

or $$(D + aD')(D - aD' + 2ab)z = 0 \qquad \ldots(1)$$

Here, we see that equation (1) and hence the given partial differential equation is a **reducible non-homogeneous linear partial differential equation.**

The part complementary function of C.F. corresponding to the factor $(D + aD')$ is $\phi_1(y - ax)$.

Again, the part of complementary function C.F. corresponding to the factor $(D - aD' + 2ab)$ is $e^{-2abx}\phi_2(y + ax)$

Hence, the general solution of (1) is given by

$$z = \phi_1(y - ax) + e^{-2abx}\phi_2(y + ax) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Example 3. Solve $r + 2s + t + 2p + 2q + z = 0$.

Solution. The given partial differential equation can be written as

$$\left(\frac{\partial^2 z}{\partial x^2}\right) + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial^2 z}{\partial y^2}\right) + 2\left(\frac{\partial z}{\partial x}\right) + 2\left(\frac{\partial z}{\partial y}\right) + z = 0$$

or $$(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$$

or $$[(D + D')^2 + 2(D + D') + 1]z = 0$$

or $$(D + D' + 1)^2 z = 0 \qquad \dots(1)$$

Here, we see that equation (1) is a **reducible non-homogeneous linear partial differential equation** and there are two repeated linear factors $(D + D' + 1)$.

∴ The required general solution is given by

$$z = e^{-x}[\phi_1(y - x) + x\phi_2(y - x)] \qquad \dots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

EXERCISE

Solve the following partial differential equations:

1. $(D - D' + 1)(D + 2D' - 3)z = 0$
2. $(DD' + aD + bD' + ab)z = 0$
3. $r + 2s + t + 2p + 2q + z = 0$
4. $(D + 1)(D + D' - 1)z = 0$
5. $(D^2 - D'^2 + D - D')z = 0$
6. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$
7. $(D^2 - DD' + D' - 1)z = 0$
8. $(D^2 + DD' + D' - 1)z = 0$

ANSWERS

1. $z = e^{-x}\phi_1(y + x) + e^{2x}\phi_2(y - 2x)$
2. $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$
3. $z = e^{-x}\{\phi_1(y - x) + x\phi_2(y - x)\}$
4. $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$
5. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$

8.3 Linear Differential equations

Linear differential equations are a fundamental concept in mathematics and physics, particularly in the study of dynamical systems and modeling natural phenomena. A linear differential equation is an equation that is linear in the unknown function and its derivatives. It can be expressed in the general form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

Where y is the unknown function of x, y′ denotes the derivative of y with respect to x, and g(x) is a given function of x. The coefficients $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ can be functions of x but are assumed to be continuous.

Types of Linear Differential Equations

Ordinary Linear Differential Equations (ODEs)

In the case where the unknown function depends on a single independent variable, x, the differential equation is called an ordinary linear differential equation (ODE). Ordinary differential equations arise in many areas of science and engineering, describing phenomena ranging from population growth to the behavior of electrical circuits.

Example:

$$y'' + p(x)y' + q(x)y = g(x)$$

Partial Linear Differential Equations (PDEs)

When the unknown function depends on multiple independent variables, the equation becomes a partial linear differential equation (PDE). These are often used to model physical systems where the behavior of the system depends on multiple spatial or temporal dimensions.

Example

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Methods for Solving Linear Differential Equations

1. **Method of Undetermined Coefficients:** This method is applicable to linear differential equations with constant coefficients and a right-hand side that is a

polynomial, exponential, sine, cosine, or a linear combination of these functions.

2. **Variation of Parameters:** This method is used for solving non-homogeneous linear differential equations. It involves finding a particular solution by varying the parameters in the general solution of the corresponding homogeneous equation.
3. **Method of Integrating Factors:** This method is used to solve first-order linear ordinary differential equations. It involves multiplying both sides of the equation by an integrating factor to make the left-hand side a perfect differential, thus simplifying the integration process.

Solved examples:

1 . Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Sol: Given equation is $(1+y^2)\frac{dx}{dy} = (\tan^{-1}y - x)$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right) . x = \frac{\tan^{-1} y}{1+y^2}$$

It is the form of $\frac{dx}{dy} + p(y).x = Q(y)$

$$\text{I.F} = e^{\int p(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$$\Rightarrow \text{General solution is} \quad x.\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + c$$

$$\Rightarrow x.\, e^{\tan^{-1} y} = \int t .\, e^t\, dt + c$$

[put $\tan^{-1} y = t$

$\Rightarrow \frac{1}{1+y^2} dy = dt$]

$$\Rightarrow x.\, e^{\tan^{-1} y} = t\,.\, e^t - e^t + c$$

$$\Rightarrow x.\, e^{\tan^{-1} y} = \tan^{-1} y . e^{\tan^{-1} y} - e^{\tan^{-1} y} + c$$

$$\Rightarrow x = \tan^{-1} y - 1 + c/e^{\tan^{-1} y} \text{ is the required solution}$$

2. Solve (x+y+1)$\frac{dy}{dx}$ = 1.

Sol: Given equation is (x+y+1)$\frac{dy}{dx}$ = 1.

$=> \frac{dx}{dy} - x = y+1.$

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

$=> I.F = e^{\int p(y)dy} = e^{-\int dy} = e^{-y}$

General solution is x X I.F = $\int Q(y) \times I.F.dy + c$

$=> x . e^{-y} = \int (1+y)\, e^{-y} dy + c$

$=> x . e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + c$

$=> \quad x e^{-y} = - e^{-y} - y\, x e^{-y} . e^{-y} + c$

$=> \quad x e^{-y} = - e^{-y}(2+y) + c$ .//

3. Solve $y^1 + y = e^{e^x}$

Sol: Given equation is $y^1 + y = e^{e^x}$

It is of the form $\frac{dy}{dx} + p(x).y = \emptyset(x)$

Where p(x) =1 Q(x) = e^{e^x}

$=> \quad I.F = e^{\int p(x)dx} = e^{\int dx} = e^x$

General solution is y x I.F = $\int Q(x) \times I.F.dx + c$

$=> \quad y.\ e^x = \int e^{e^x} e^x dx + c$

$=> y.\ e^x = \int e^t dt + c$

$=> y.\ e^x = e^t + c$

$=> y.\ e^x = e^{e^x} + c$

$\left\{ \begin{array}{l} \text{put } e^x = t \\ e^x dx = dt \end{array} \right\}$

8.4 Summary

Understanding whether a differential equation is homogeneous or non-homogeneous is crucial for selecting the appropriate solution technique. Homogeneous equations have solutions that are directly related to the structure of the equation, while non-homogeneous equations require additional terms to be considered, making their solutions more complex.

8.5 Keywords

- Linear
- Constant coefficients
- Characteristic roots
- Zero right-hand side
- Non-homogeneous
- Non-linear
- Particular solution

8.6 Self-Assessment Questions

Q1. How are the solutions of homogeneous linear differential equations affected by changes in initial conditions?

Q2. Can you explain the significance of characteristic roots in solving homogeneous linear differential equations?

Q3. What methods can be employed to solve homogeneous differential equations with variable coefficients?

Q4. What role does the method of undetermined coefficients play in solving non-homogeneous differential equations?

Q5. What distinguishes non-homogeneous linear differential equations from their homogeneous counterparts?

8.7 Case Study:

Imagine you are tasked with modeling the growth of a population of rabbits in a controlled environment, such as a wildlife reserve. The population dynamics of the rabbits are influenced by various factors, including birth rate, death rate, and interactions with the environment.

Question: Decide to model the population of rabbits using a differential equation, considering both homogeneous and non-homogeneous scenarios to account for different factors.

8.8 References

1. Grewal . B.S., “Elementary Engineering Mathematics”, Khanna publications 34th Ed., 2015.
2. Gupta, S. P and Kapoor V.K, Fundamental of Mathematical Statistics, Sultan Chand and Sons, New Delhi.

UNIT : 9

CENTRAL TENDENCY

Learning Objectives:

- To understand Measure of central tendency
- To understand GM and H.M.
- To understand weighted mean form, quartile, deciles and percentiles

Structure:

9.1 Measure of central tendency

9.2 GM and H.M and weighted mean form

9.3 Quartile, deciles and percentiles

9.4 Summary

9.5 Keywords

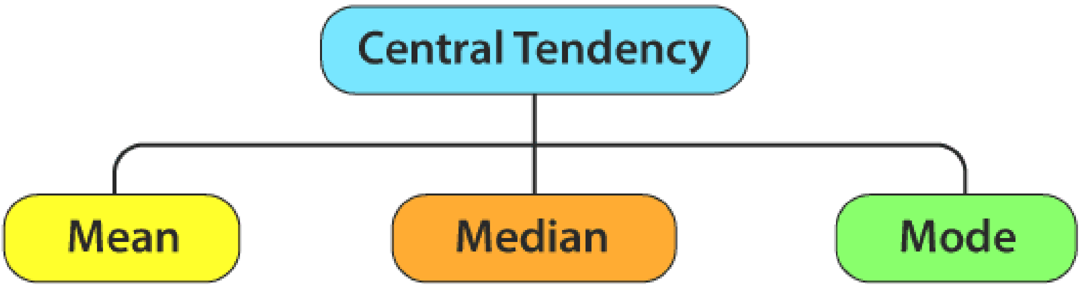
9.6 Self-Assessment Questions

9.7 Case Study

9.8 References

9.1 Measure of central tendency

A simple flowchart to define measures of central tendency in term of Arithmetic Mean , mode and median.

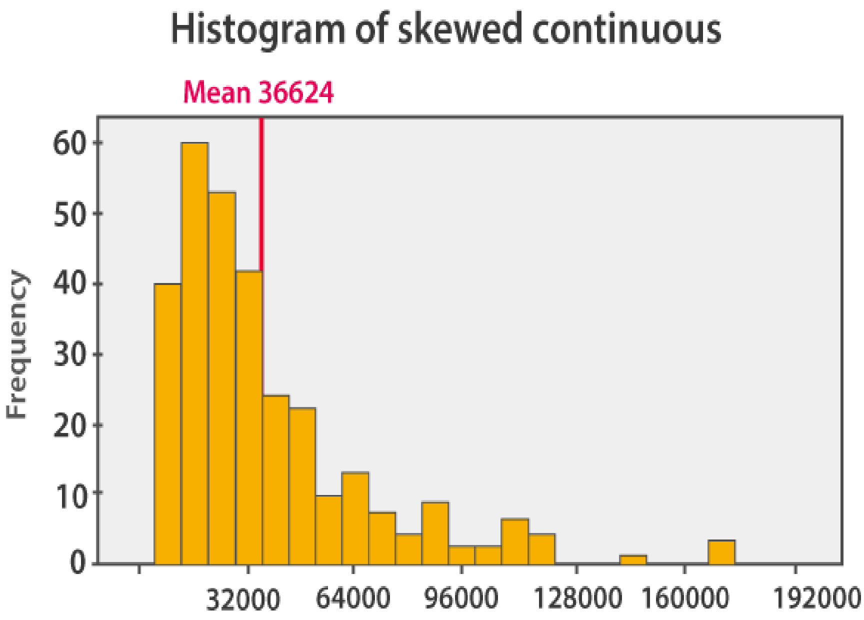


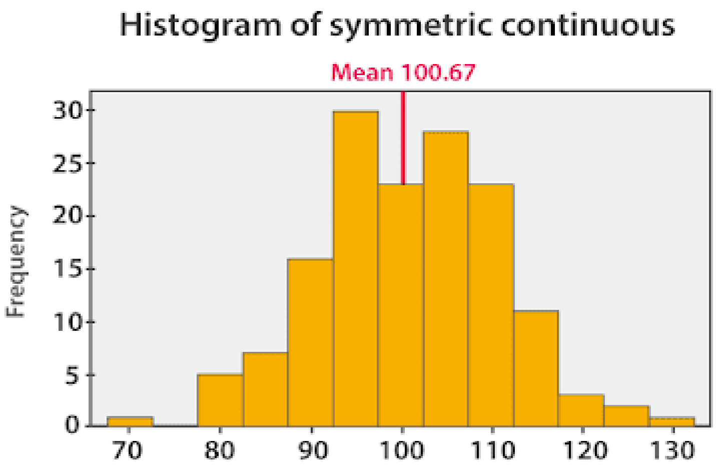
Arithmetic Mean:

- The arithmetic mean is average value of all observations (x_1, x_2. x_3,…).
- It's sensitive to outliers and extreme values.

$$Mean = \frac{x_1 + x_2 + .. + x_n}{n}$$

The mean value of both skewed and symmetric continuous data is shown in the histogram below.





Median:

- Middle value of any statistical data by putting ascending or descending order is known as median .

Examples of datasets with their medians presented in tabular form:

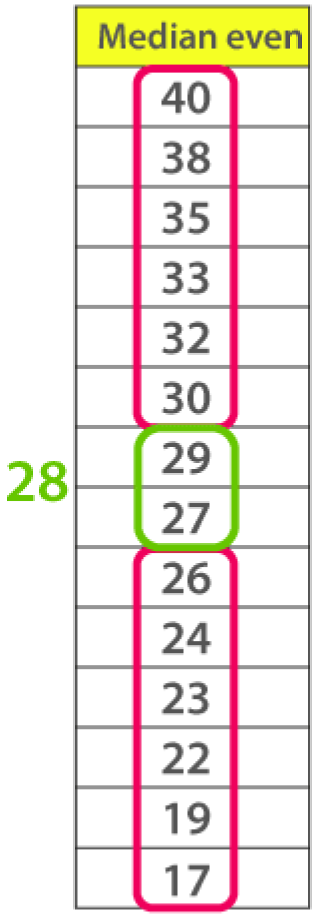
Dataset	Median
2, 5, 7, 9, 12	7
3, 5, 6, 8, 10, 12	6.5
5, 5, 7, 8, 9, 10	7.5
1, 2, 3, 4	2.5
10, 20, 30, 40	25

Examine the dataset that has the odd number of observations (23, 21, 18, 16, 15, 13, 12, 10, 9, 7, 6, 5, and 2) sorted in descending order.

Median odd
23
21
18
16
15
13
12
10
9
7
6
5
2

Median = 12

Examine the dataset that has the odd number of observations (40, 38, 35, 33, 32, 30, 29, 27, 26, 24, 23, 22, 19, and 17) sorted in descending order.



Median = $\frac{29+27}{2} = 28$

Mode: A number which repeats more times in any observations is known as mode.

These measures provide different perspectives on the central tendency of data and are used depending on the characteristics of the dataset and the specific requirements of the analysis.

Examine the dataset that is provided: 5, 4, 2, 3, 2, 1, 5, 4, 5.

Mode
5
5
5
4
4
3
2
2
1

Mode= 5

9.2 GM and H.M and weighted mean form

Arithmetic Mean:

- The arithmetic mean is average value of all observations (x_1, x_2, x_3,…).
- It's sensitive to outliers and extreme values.

$$Mean = \frac{x_1 + x_2 + .. + x_n}{n}$$

- Formula: $\text{Arithmetic Mean} = \frac{\sum_{i=1}^{n} x_i}{n}$

Geometric Mean:

- In mathematical notation of geometric mean is:
  - Formula: $\text{Geometric Mean} = \sqrt[n]{x_1 \times x_2 \times \ldots \times x_n}$

Where X_1, X_2, X_3….. are the different data and n is number of observations.

Harmonic Mean:

- In mathematical notation of Harmonic mean is

- Formula: $\text{Harmonic Mean} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$

Weighted Mean Form and formula

The weighted mean formula calculates the average of a dataset. In mathematical notation:

$$\text{Weighted Mean} = \frac{\sum_{i=1}^{n}(x_i \times w_i)}{\sum_{i=1}^{n} w_i}$$

Where:

1. xirepresents the value of the ith observation.
2. represents the weight corresponding to the ith observation.
3. $n$is the total number of observations.

9.3 Quartile, Deciles and Percentiles

Quartiles

The quartiles are denoted as Q1, Q2, and Q3. Q2 is also known as the median.

Quartiles Formula:

1. **Q1 (First Quartile):**
   - $Q1 = \text{Lower quartile} = 25\text{th percentile}$
   - Formula: $Q1 = \text{Value at } \left(\frac{n+1}{4}\right) \text{ th position}$
   - When the position is not an integer, you can take the average of the values at the floor and ceil positions.
2. **Q2 (Second Quartile, Median):**
   - $Q2 = \text{Median} = 50\text{th percentile}$
   - Formula: $Q2 = \text{Value at } \left(\frac{n+1}{2}\right) \text{ th position}$
   - When the position is not an integer, you can directly take the value at the position.

- First Quartile (Q1): (1/4) * (n + 1)
- Second Quartile (Q2, Median): (2/4) * (n + 1)
- Third Quartile (Q3): (3/4) * (n + 1)

Examples : 1: Determine the following data's quartiles: 2, 3, 6, 7, 8, 10, 23, and 34.

Solution:

To find the quartiles of the given data set {4, 6, 7, 8, 10, 23, 34}, we can follow these steps:

1. Arrange the data in ascending order:
   4, 6, 7, 8, 10, 23, 34

2. Calculate the position of the quartiles:

   - First Quartile (Q1): (1/4) * (n + 1)
   - Second Quartile (Q2, Median): (2/4) * (n + 1)
   - Third Quartile (Q3): (3/4) * (n + 1)

   Where 'n' is the total number of data points, which in this case is 7.

3. Calculate the quartiles:

   - Q1: (1/4) * (7 + 1) = (1/4) * 8 = 2
     The value at the 2nd position is 6. Therefore, Q1 = 6.
   - Q2 (Median): (2/4) * (7 + 1) = (2/4) * 8 = 4
     Since the number of data points is odd, the median is the value at the (4th + 1) / 2 = 5/2 = 2.5th position.
     The value at the 2.5th position is the average of the 2nd and 3rd values, which is (6 + 7) / 2 = 6.5. Therefore, Q2 = 6.5.
   - Q3: (3/4) * (7 + 1) = (3/4) * 8 = 6
     The value at the 6th position is 23. Therefore, Q3 = 23.

Deciles

The decile method is employed to divide a distribution into ten equal parts. Each data point is assigned a decile rank, allowing for sorting the data in ascending or descending order. Unlike quartiles, which consist of four categorical buckets, and percentiles, which have 100, deciles comprise ten categorical buckets.

The concept of deciles finds extensive use in finance and economics for data analysis purposes. It proves valuable in evaluating portfolio performance within the finance domain. In this article, we delve into the definition and rank of deciles, while also exploring examples that illustrate how to calculate decile values.

Decile Formula:

Ungrouped Data : $D(x) = (n + 1) * \frac{x}{10}$

Grouped Data : $D(x) = l + \frac{w}{f}\left(\frac{Nx}{10} - C\right)$

Percentiles

A percentile is a measure that indicates how a particular score compares to other scores within the same dataset.

Formula

$$P_x = \frac{x(n + 1)}{100}$$

P_x = The value at which x percentage of data lie below that value

n = Total number of observations

9.4 Summary

An realistic depiction of the average is given by the weighted mean ,when certain values in the dataset are more significant or carry more weight than others. It's frequently used to account for variations in the significance of data points in a variety of sectors, including finance, economics, and education.

The docile method is employed to divide a distribution into ten equal parts. Each data point is assigned a decile rank, allowing for sorting the data in ascending or descending order. Unlike quartiles, which consist of four categorical buckets, and percentiles, which have 100, deciles comprise ten categorical buckets.

A percentile is a measure that indicates how a particular score compares to other scores within the same dataset. Although the definition of a percentile may vary, Decile is commonly interpreted as the percentage of values in a dataset that fall below a particular specified value

Keywords

- Arithmetic mean
- Mode
- Median
- Mode
- Quartiles:
- Percentiles.

9.6 Self-Assessment Questions

1. Which measure of central tendency is most appropriate for skewed datasets?
2. For data that are favourably skewed, what is the optimal quality of a good measure of central tendency?
3. Determine the median for data set: 10, 12, 14, 16, 18.
4. Explain the concept of quartiles and their significance in analysing datasets.
5. Explain the concept of mean , median and mode.

9.7 Case Study

A school administration wants to evaluate the performance of its students in a particular subject. They want to comprehend the students’ grades and choose the measure of central tendency that best represents the data. You have been tasked as a statistician with analysing the scores, Identifying quartiles, deciles, and percentiles as well as computing a number of central tendency measures, including mean, median, mode, geometric mean, harmonic mean, and weighted mean. The scores obtained by a group of students in a subject are as follows: 72, 65, 80, 75, 85, 70, 90, 80, and 78.

Questions:

1. Calculate the mean, median, and mode of the scores.
2. The school administration wants to identify students' performance at specific percentile levels. They are particularly interested in the 80th percentile and the 60th percentile. Determine the score at the 80th percentile using the given data.
3. Calculate the interquartile range (IQR) of the scores and identify the score at the 60th percentile within the IQR range.

9.8 References

1. Grewal . B.S., “Elementary Engineering Mathematics”, Khanna publications 34th Ed., 2020.
2. Gupta, S. P and Kapoor V.K, Fundamental of Mathematical Statistics, Sultan Chand and Sons, New Delhi.